Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 259 (2015) 2121–2157

Journal of Differential Equations

www.elsevier.com/locate/jde

Classification and sharp range of flux-pairs for radial solutions to a coupled system

Zhi-You Chen [*,1], Yong-Li Tang [2]

Department of Mathematics, National Central University, Chung-Li 32001, Taiwan

Received 10 February 2015; revised 23 March 2015

Available online 3 April 2015

Abstract

In this paper, we study a coupled system of two nonlinear partial differential equations in the plane, which is related to Liouville equations, non-abelian Higgs BPS vortex equations or two Higgs electroweak model, with singularities at the origin. In addition to deriving the uniqueness of the so-called *topological solutions*, we also clarify the structure of all types of solutions, including the blow-up ones, under various conditions on coefficients and parameters appearing in the system. Furthermore, the sharp range of flux-pairs associated with specific types of solutions is considered as well.

MSC: primary 35J47; secondary 35A20

Keywords: Classification of solutions; Sharp range of flux-pairs; Coupled system

* Corresponding author.

E-mail addresses: zhiyou@math.ncu.edu.tw (Z.-Y. Chen), tangyl@math.ncu.edu.tw (Y.-L. Tang).

1 Work partially supported by Ministry of Science and Technology, Taiwan (No. MOST-103-2115-M-008-011-MY3), the National Natural Science Foundation of China (No. 11401144) and National Center for Theoretical Sciences of Taiwan.

2 Work partially supported by National Center for Theoretical Sciences of Taiwan.

http://dx.doi.org/10.1016/j.jde.2015.03.021

0022-0396/

1. Introduction

In this article, we study solutions of the following coupled system

$$\begin{cases} \Delta u = ae^u + be^v - \gamma_1 + 4\pi N_1 \delta_O \\ \Delta v = be^u + ce^v - \gamma_2 + 4\pi N_2 \delta_O \end{cases} \text{in } \mathbf{R}^2, \tag{1.1}$$

where $\Delta = \Sigma_{i=1}^2 \partial^2/\partial x_i^2$, O is the origin point of $\mathbf{R}^2$, and δ_p is the Dirac measure at p for $p \in \mathbf{R}^2$; a, $-b$ and c are positive constants; N_1 and N_2 are non-negative constants; γ_1 and γ_2 are real numbers, and $(a, b, c, \gamma_1, \gamma_2)$ satisfies

$$c\gamma_1 - b\gamma_2 \geq 0 \text{ and } a\gamma_2 - b\gamma_1 \geq 0. \tag{1.2}$$

To our knowledge for this kind of systems, (1.1) without Dirac sources was studied for the first time by Chanillo and Kiessling in [18] for $\gamma_1 = \gamma_2 = 0$. To cite other sources of (1.1), we consider the Bennett system for 1-component plasma:

$$\Delta\phi + 4\pi \frac{Dq}{Z} e^{-\beta q(\phi - \nu\psi)} = 0, \quad \Delta\psi + 4\pi \frac{Dq\nu}{Z} e^{-\beta q(\phi - \nu\psi)} = 0 \tag{1.3}$$

for some specific parameters D, β and ν, and

$$Z = Z[\phi, \psi] = \int_{\mathbf{R}^2} e^{-\beta q[\phi(\mathbf{x}) - \nu\psi(\mathbf{x})]} d\mathbf{x}.$$

In the high temperature or low density limit, a semi-conformal system of Bennett equations like (1.3) is a reduced form of a nonlinear system of so-called finite-temperature Thomas–Fermi equations, which describes the relativistic quantum mechanics of a stationary beam of counter-streaming, negatively charged electrons and one species of positively charged ions in the semi-classical limit. Bennett equations also constitute a Liouville-type system, which is generally associated with an asymmetric coefficient matrix with some negative entries and always rank 2. See, *e.g.*, [5,9,10] for details. We remark that if $b^2 \neq ac$, then (1.3) is equivalent to (1.1) with $N_1 = N_2 = \gamma_1 = \gamma_2 = 0$.

Another motivation for studying coupled equations related to (1.1) arises from the non-abelian Higgs BPS vortex equations (see, *e.g.*, [1,12,15,19])

$$\begin{cases} \Delta(u + v) = \theta(e^u + e^v - 2) + 4\pi \sum_{s=1}^n \delta_{p_s} \text{ in } \mathbf{R}^2, \\ \Delta(u - v) = \eta(e^u - e^v) + 4\pi \sum_{s=1}^n \delta_{p_s} \text{ in } \mathbf{R}^2, \end{cases} \tag{1.4}$$

where $\theta, \eta > 0$ and $n \in \mathbf{N}$. If we treat a specific solution $(\hat{u}, \hat{v})$ of (1.4) with repeated points $p_1 = \cdots = p_n = p$, then $\hat{u}$ and $\hat{v}$ must satisfy

$$\begin{cases} \Delta\hat{u} = \frac{\theta+\eta}{2} e^{\hat{u}} + \frac{\theta-\eta}{2} e^{\hat{v}} - \theta + 4\pi n \delta_p \text{ in } \mathbf{R}^2, \\ \Delta\hat{v} = \frac{\theta-\eta}{2} e^{\hat{u}} + \frac{\theta+\eta}{2} e^{\hat{v}} - \theta \text{ in } \mathbf{R}^2. \end{cases}$$