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A note on the Camassa–Holm equation

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Abstract

We consider a generalized Camassa–Holm equation. We improve the compactness argument of [5–7,17], establishing the existence of global weak solutions in H^1 without the use of an Oleĭnik-type estimate. The potential interest of this improvement is linked to convergence analysis of numerical schemes, for which it may not be easy to verify an Oleĭnik-type estimate.

MSC: 35G25; 35L05; 35A05

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1. Introduction

We are interested in the Cauchy problem for the nonlinear equation

$$\begin{cases} \partial_t u - \partial^3_{txx} u + \partial_x \left(\frac{g(u)}{2}\right) = \gamma \left(2\partial_x u \partial^2_{xx} u + u\partial^3_{xxx} u\right), & t > 0,\ x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where the constant $\gamma \in \mathbb{R}$ and the function $g : \mathbb{R} \to \mathbb{R}$ are given. Observe that if $\gamma = 1$ and $g(u) = 3u^2$, then (1.1) is the classical Camassa–Holm equation [3,16]. Dai [11,10,12] derived (1.1) as an equation describing finite length, small amplitude radial deformation waves in cylindrical compressible hyperelastic rods. The constant γ is given in terms of the material constants and the prestress of the rod.

We shall assume

$$u_0 \in H^1(\mathbb{R}), \qquad g \in C^\infty(\mathbb{R}), \qquad g(0) = 0, \qquad \gamma > 0.$$

Observe that the case $\gamma = 0$ is much simpler than the one we are considering. Moreover, if $\gamma < 0$ we can use a similar argument.

The well-posedness of the Camassa–Holm wave equation under only H^1 regularity on the initial condition has been widely studied, see for example [1,2,5,13–15,17,18]. The relevance of the H^1 space is that its norm is preserved (up to an inequality) by the solution operator, and H^1 is the minimal regularity needed to make distributional sense to (1.1) via (1.2) below. This is also consistent with the fact that (1.1) experiences wave breaking, in the sense that solutions remain bounded while their x-derivative may blow up [8,9].

Rewriting Eq. (1.1) as

$$(1 - \partial_x^2)\partial_t u + \gamma(1 - \partial_x^2)(u\partial_x u) + \partial_x \left(h(u) + \frac{\gamma}{2}(\partial_x u)^2\right) = 0,$$

where

$$h(u) := \frac{g(u) - \gamma u^2}{2},$$

we see that Eq. (1.1) formally is equivalent to the elliptic–hyperbolic system

$$\partial_t u + \gamma u \partial_x u + \partial_x P = 0, \qquad -\partial^2_{xx} P + P = h(u) + \frac{\gamma}{2}(\partial_x u)^2. \tag{1.2}$$

Moreover, since $e^{-|x|}/2$ is the Green's function of the Helmholtz operator $-\partial^2_{xx} + 1$, we have that

$$P(t, x) = \frac{1}{2}\int_{\mathbb{R}} e^{-|x-y|}\left(h(u(t, y))^2 + \frac{\gamma}{2}(\partial_x u(t, y))^2\right) dy,$$

$$\partial_x P(t, x) = \frac{1}{2}\int_{\mathbb{R}} e^{-|x-y|}\operatorname{sign}(y - x)\left(h(u(t, y))^2 + \frac{\gamma}{2}(\partial_x u(t, y))^2\right) dy.$$