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On the wave length of smooth periodic traveling waves of the Camassa–Holm equation ☆

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Abstract

This paper is concerned with the wave length λ of smooth periodic traveling wave solutions of the Camassa–Holm equation. The set of these solutions can be parametrized using the wave height a (or "peak-to-peak amplitude"). Our main result establishes monotonicity properties of the map $a \longmapsto \lambda(a)$, i.e., the wave length as a function of the wave height. We obtain the explicit bifurcation values, in terms of the parameters associated with the equation, which distinguish between the two possible qualitative behaviors of $\lambda(a)$, namely monotonicity and unimodality. The key point is to relate $\lambda(a)$ to the period function of a planar differential system with a quadratic-like first integral, and to apply a criterion which bounds the number of critical periods for this type of systems.

MSC: 35Q35; 34C25

Keywords: Camassa–Holm equation; Traveling wave solution; Wave length; Wave height; Center; Critical period

1. Introduction and main result

The Camassa–Holm (CH) equation

$$u_t + 2\kappa\, u_x - u_{txx} + 3\,u\,u_x = 2\,u_x u_{xx} + u\,u_{xxx}, \qquad x \in \mathbb{R},\ t > 0, \tag{1}$$

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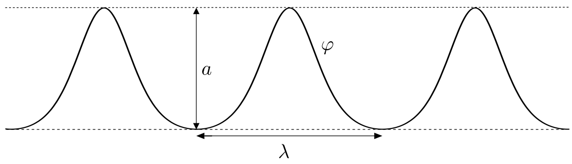

Fig. 1. Smooth periodic TWS φ of CH with wave length λ and wave height a.

arises as a shallow water approximation of the Euler equations for inviscid, incompressible and homogenous fluids propagating over a flat bottom, where $u(x,t)$ describes the horizontal velocity component and $\kappa \in \mathbb{R}$ is a parameter related to the critical shallow water speed. This equation was first derived by Fokas and Fuchssteiner [20] as an abstract bi-Hamiltonian equation with infinitely many conservation laws, and later re-derived by Camassa and Holm [4] from physical principles. For a discussion on the relevance and applicability of the CH equation in the context of water waves we refer the reader to Johnson [29–31] and more recently Constantin and Lannes [13]. We point out that for a large class of initial conditions the CH equation is an integrable infinite-dimensional Hamiltonian system [1,7,8,14,15,30], and it is known that the solitary waves of CH are solitons which are orbitally stable [15,19]. The smooth periodic traveling wave solutions are orbitally stable as well [34]. Some classical solutions of the CH equation develop singularities in finite time in the form of wave breaking: the solution remains bounded but its slope becomes unbounded [5,7,11,12,18,36,38]. After blow-up the solutions can be recovered in the sense of global weak solutions, see [2,3] and also [27,25].

In the present paper, we consider traveling wave solutions of the form

$$u(x,t) = \varphi(x - ct), \tag{2}$$

for $c \in \mathbb{R}$ and some function $\varphi : \mathbb{R} \to \mathbb{R}$. We denote $s = x - ct$ the independent variable in the moving frame. Inserting the Ansatz (2) into Eq. (1) and integrating once we obtain the corresponding equation for traveling waves,

$$\varphi''(\varphi - c) + \frac{(\varphi')^2}{2} + r + (c - 2\kappa)\,\varphi - \frac{3}{2}\varphi^2 = 0, \tag{3}$$

where $r \in \mathbb{R}$ is a constant of integration and the prime denotes derivation with respect to s. A solution φ of (3) is called a *traveling wave solution* (TWS) of the Camassa–Holm equation (1). Lenells [35] provides a complete classification of all (weak) traveling wave solutions of the Camassa–Holm equation. In the present paper, we focus on *smooth periodic* TWS of the Camassa–Holm equation, which can be shown to have a unique maximum and minimum per period, see [35]. In the context of fluid dynamics the period of such a solution is called *wave length*, which we will denote by λ. The difference between the maximum (wave crest) and the minimum (wave trough) is called *wave height*, see Fig. 1, which we will denote by a (in some contexts this quantity is also called “peak-to-peak amplitude”).

The aim of this paper is to study the dependence of the wave length λ of smooth periodic TWS of the Camassa–Holm equation (1) on their wave height a. Our main result shows that $\lambda(a)$ is a well-defined function and that it is either monotonous or unimodal. More precisely:

Theorem A. *Given c, κ with $c \neq -\kappa$, there exist real numbers $r_1 < r_{b_1} < r_{b_2} < r_2$ such that the differential equation (1) has smooth periodic TWS of the form (2) if, and only if, the integration*