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Generalized Morrey estimates for the gradient of divergence form parabolic operators with discontinuous coefficients ☆

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Abstract

We consider the Cauchy–Dirichlet problem for linear divergence form parabolic operators in bounded Reifenberg-flat domains. The coefficients are supposed to be only measurable in one of the space variables and small *BMO* with respect to the others. We obtain boundedness of the Hardy–Littlewood maximal operator in the generalized Morrey spaces $W^{p,\varphi}$, $p \in (1,\infty)$ and weight φ satisfying certain supremum condition. This permits us to obtain Calderón–Zygmund type estimate for the gradient of the weak solution of the problem.

MSC: primary 35K20; secondary 35B65, 35K40, 35R05, 42B25, 46E30

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1. Introduction

The classical *Morrey spaces* $L^{p,\lambda}$ are originally introduced by Morrey in [25] in order to prove local Hölder continuity of the solutions to elliptic systems of partial differential equations. A real valued function f is said to belong to the Morrey space $L^{p,\lambda}$ with $p \in (1,\infty)$, $\lambda \in (0,n)$ provided the following norm is finite

$$\|f\|_{L^{p,\lambda}(\mathbb{R}^n)} = \left(\sup_{(x,r)\in\mathbb{R}^n\times\mathbb{R}_+} \frac{1}{r^\lambda} \int_{\mathcal{B}_r(x)} |f(y)|^p \, dy \right)^{1/q}$$

where the supremum is taken over all balls $\mathcal{B}_r(x) \subset \mathbb{R}^n$. The main result connected with these spaces is the following celebrated lemma: let $|Df| \in L^{p,n-\lambda}$ even locally, with $n-\lambda < p$, then u is Hölder continuous of exponent $\alpha = 1 - \frac{n-\lambda}{p}$. This result has found many applications in the study of the regularity of the solutions to elliptic and parabolic equations and systems. In [11] Chiarenza and Frasca showed boundedness of the *Hardy–Littlewood maximal operator* in $L^{p,\lambda}(\mathbb{R}^n)$ that allows them to prove continuity in these spaces of some classical integral operators.

In [24] Mizuhara extended Morrey's concept taking a weight function $\varphi(x,r) : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ instead of r^λ. So was put the beginning of the study of the *generalized Morrey spaces* $M^{p,\varphi}$, $p > 1$ with φ belonging to various classes of weight functions. In [26] Nakai proved boundedness of the maximal and Calderón–Zygmund operators in $M^{p,\varphi}$ imposing suitable integral and doubling conditions on φ. These results allow to study the regularity of the solutions of various linear elliptic and parabolic boundary value problems in $M^{p,\varphi}$ (see [8,28,30]). A further development of the generalized Morrey spaces can be found in the works of Guliyev [15–17], see also [1,18–22] and the references therein. Here we consider a supremum condition on the weight (14) which is optimal and ensure the boundedness of the Hardy–Littlewood maximal operator in $M^{p,\varphi}$. We use this maximal inequality to obtain the Calderón–Zygmund type estimate for the gradient of the solution of the problem (1) in the $M^{p,\varphi}$.

The presented here result is a natural extension of the previous papers of Byun, Palagachev and Wang [7] which deals with the regularity problem for parabolic equations in classical Lebesgue classes and of Byun, Palagachev and Softova [5,6] where the problem (1) is studied in the framework of the weighted Lebesgue and Orlicz spaces with a Muckenhoupt weight and the classical Morrey spaces $L^{p,\lambda}(Q)$ with $\lambda \in (0, n+2)$. See also the recent results of Byun [3,4], Byun and Wang [9], Byun and Softova [8], Dong and Kim [14] and Dong [13]. In our previous works we also studied the global regularity in $M^{p,\varphi}$ of strong solutions of various boundary value problems for linear elliptic and parabolic equations with *VMO* (or small *BMO*) coefficients. In those works we used explicit representation formula for the solutions and the boundedness in $M^{q,\varphi}$ of certain integral operators (see [21,22,29]). Here we extend these study, obtaining regularity estimates for the gradient of the weak solutions of boundary value problems with the Dirichlet data for divergence form linear operators.

The our main tool is the maximal inequality in $M^{p,\varphi}$ obtained in Section 3 and a version of the Vitali covering lemma.