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Perturbation theory of a symmetric center within Liénard equations ☆

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Abstract

In this article, we introduce the use of Lambert function to develop further the global perturbation theory of an integrable Liénard equation which displays a symmetric center. We prove a global Morse lemma for the first integral and deduce the existence of an associated Picard–Fuchs system. We revisit previous contributions to first-order perturbation theory with the help of these new analytic techniques and in particular, we check that the fundamental integrals are linearly independent. The Lambert function allows to find an expansion formula for these integrals. We also study the possibility to develop a higher-order perturbation theory. The algorithm of the successive derivatives works in general in the class of analytic functions on the domain D where the level sets of the first integral are ovals. We end the article with some results on the first integral of a symmetric Liénard equation deduced from the algorithm of successive derivatives.

MSC: primary 34C05, 34A34, 34C14

Keywords: Liénard systems; Abelian integrals; Lambert function

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1. Introduction

The simplified version of the second part of Hilbert's 16th problem is related with the least upper bound on the number of limit cycles of the following Liénard system of type (m, n)

$$\begin{cases} \dfrac{dx}{dt} = y - F(x), \\ \dfrac{dy}{dt} = -g(x), \end{cases} \tag{1}$$

where $g(x)$ and $f(x) = \frac{dF(x)}{dx}$ are polynomials with degree m and n, respectively. In 1977 A. Lins, W. de Melo and C.C. Pugh in [17] studied this version and stated a conjecture: *if $g(x) = x$, then system (1) has at most $[\frac{n}{2}]$ limit cycles, where $[\frac{n}{2}]$ means the largest integer less than or equal to $\frac{n}{2}$*. The conjecture is true for $n = 1, 2$ in [17]. Recently, C. Li and J. Llibre proved that the conjecture is true for $n = 3$ in [15]. And for $n = 6$ this conjecture is not true as Dumortier, Panazzolo and Roussarie in [2] proposed a counter-example based on singular perturbation theory. The conjecture for $n = 4, 5$ is still open. For more information about the simplified version of Hilbert's 16th problem and related topics see [19] and [21]. On the other hand, there have been many results on the number of limit cycles of system (1) by considering limit cycles which bifurcate from a center or focus, see [11,18] and references therein.

In the present paper, one of our purposes is to consider the number of limit cycles which can bifurcate from the periodic orbits of a nonlinear center perturbed inside the class of generalized polynomial Liénard system of type (m, n). We consider the following integrable system with a nonlinear center

$$\begin{cases} \dfrac{dX}{dt} = Y - aX^2, \\ \dfrac{dY}{dt} = -X, \end{cases} \tag{2}$$

where $a \neq 0$, which is a constant. It is easy to check that $(0, 0)$ is a center of system (2). By scaling $X = \frac{x}{2a}$ and $Y = \frac{y}{2a}$, we can transform (2) to

$$\begin{cases} \dfrac{dx}{dt} = y - \dfrac{1}{2}x^2, \\ \dfrac{dy}{dt} = -x. \end{cases} \tag{3}$$

System (3) has a first integral

$$H(x, y) = e^{-y}(\frac{1}{2}x^2 - y - 1) = h, \quad -1 < h < 0 \tag{4}$$

with an integrating factor taken as exponential function

$$\mu(y) = e^{-y}. \tag{5}$$