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Regularization of hidden dynamics in piecewise smooth flows

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Abstract

This paper studies the equivalence between differentiable and non-differentiable dynamics in $\mathbb{R}^n$. Filippov's theory of discontinuous differential equations allows us to find flow solutions of dynamical systems whose vector fields undergo switches at thresholds in phase space. The canonical *convex combination* at the discontinuity is only the linear part of a *nonlinear combination* that more fully explores Filippov's most general problem: the differential inclusion. Here we show how recent work relating discontinuous systems to singular limits of continuous (or *regularized*) systems extends to nonlinear combinations. We show that if sliding occurs in a discontinuous systems, there exists a differentiable slow–fast system with equivalent slow invariant dynamics. We also show the corresponding result for the *pinching* method, a converse to regularization which approximates a smooth system by a discontinuous one.

Keywords: Sliding; Discontinuous; Hidden; Regularization; Pinching; Singular perturbation

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Consider an ordinary differential equation in $x \in \mathbb{R}^n$ with a discontinuous right-hand side,

$$\dot{x} = \begin{cases} f^+(x) & \text{if} \quad h(x) > 0, \\ f^-(x) & \text{if} \quad h(x) < 0, \end{cases} \tag{1}$$

where f^+ and f^- are continuous vector fields, and h is a differentiable scalar function whose gradient ∇h is well-defined and non-vanishing everywhere. Throughout this paper we consider an open region $x \in D$ in which (1) holds. The set $\Sigma = \{x \in \mathcal{D} : h(x) = 0\}$ is called the *switching manifold*, and the regions either side of it are denoted as $\mathcal{R}^\pm = \{x \in \mathcal{D} : h(x) \gtrless 0\}$.

The term 'hidden dynamics' refers to what happens on Σ, specifically to behaviors governed by terms that disappear in $\mathcal{R}^\pm$ (hence they are 'hidden' in (1)), and which go beyond Filippov's standard theory [7]. The theory of Filippov relies heavily on two alternatives for extending (1) across $h = 0$. The first is a differential inclusion

$$\dot{x} \in \mathcal{F}(x) \qquad \text{s.t.} \quad f^+(x), f^-(x) \in \mathcal{F}(x) \tag{2}$$

which is very general because $\mathcal{F}$ is any set that contains $f^\pm$ ($\mathcal{F}$ is usually assumed to be convex to provide certain restrictions on sequences of solutions [7], but this does not prevent $\mathcal{F}$ being arbitrarily large). The second alternative is a smaller set, the convex hull of f^+ and f^-,

$$\dot{x} = Z(x;\lambda) := \frac{1+\lambda}{2} f^+(x) + \frac{1-\lambda}{2} f^-(x), \quad \lambda \in \begin{cases} \text{sign}(h(x)) & \text{if } h(x) \neq 0, \\ [-1,+1] & \text{if } h(x) = 0, \end{cases} \tag{3}$$

which is very restrictive in the sense that it selects only values of (2) that are linear combinations of $f^\pm$. Examples of the set $\mathcal{F}$ and hull $\{Z(x;\lambda) : \lambda \in [-1,+1]\}$ will be illustrated in Example 1 below, along with a third alternative that unites them.

We will refer to the transition as h changes sign in (3) as *linear switching* (implying linear dependence with respect to λ). In Filippov's theory, one seeks values of $\dot{x}$ in the sets (2) or (3) that result in continuous (though typically non-differentiable) flows at Σ. In many situations of interest, the flow obtained from (3) is unique (making possible, for example, substantial classifications of singularities and bifurcations for such systems [7,19,5]).

The problem highlighted in [10] was that between the set-valued flow of (2) and the piecewise-smooth flow of (3), a vast expanse of non-equivalent but no less valid dynamical systems can be considered. All that is lacking is a way to express them explicitly. This is provided quite simply by permitting nonlinear dependence on the transition parameter λ, in the form

$$\dot{x} = f(x;\lambda) := \frac{1+\lambda}{2} f^+(x) + \frac{1-\lambda}{2} f^-(x) + G(x;\lambda), \tag{4}$$

where

$$h(x)G(x;\lambda) = 0, \qquad \lambda \in \begin{cases} \text{sign}(h(x)) & \text{if } h(x) \neq 0, \\ [-1,+1] & \text{if } h(x) = 0, \end{cases} \tag{5}$$

with G some continuous vector field that is nonlinear in λ. An example of the set generated by $\{f(x;\lambda) : \lambda \in [-1,+1]\}$ is given in Example 1 below. We shall refer to (4) as the nonlinear