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# Differentiability of Palmer's linearization theorem and converse result for density functions ☆

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**Abstract**

We study differentiability properties in a particular case of the Palmer's linearization theorem, which states the existence of a homeomorphism $H$ between the solutions of a linear ODE system having exponential dichotomy and a quasilinear system. Indeed, if the linear system is uniformly asymptotically stable, sufficient conditions ensuring that $H$ is a $C^2$ preserving orientation diffeomorphism are given. As an application, we generalize a converse result of density functions for a nonlinear system in the nonautonomous case.


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*Keywords:* Palmer's linearization theorem; Hartman–Grobman theorem; Density functions; Preserving orientation diffeomorphism

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## 1. Introduction

The seminal paper of K.J. Palmer [19] provides sufficient conditions ensuring the topological equivalence between the solutions of the linear system

$$y' = A(t)y, \tag{1.1}$$

and the solutions of the quasilinear one

$$x' = A(t)x + f(t,x), \tag{1.2}$$

where the bounded and continuous $n \times n$ matrix $A(t)$ and the continuous function $f\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfy some technical conditions.

Roughly speaking, (1.1) and (1.2) are topologically equivalent if there exists a map $H\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ such that $\nu \mapsto H(t,\nu)$ is a homeomorphism for any fixed $t$. In particular, if $x(t)$ is a solution of (1.2), then $H[t,x(t)]$ is a solution of (1.1).

To the best of our knowledge, there are no results concerning the differentiability of the map $H$ and the purpose of this paper is to find sufficient conditions ensuring that the map above is a preserving orientation diffeomorphism of class $C^1$ (Theorem 1) and $C^2$ (Theorem 2), both under the assumption that (1.1) is uniformly asymptotically stable.

As an application of our results, we will construct a density function for the system (1.2) when $f(t,0) = 0$ (Theorem 3), generalizing a converse result in the autonomous case presented in [17].

### 1.1. Palmer's linearization theorem

We are interested in the particular case:

**Proposition 1.** *(See Palmer [19].) If the assumptions:*

- **(H1)** $|f(t,x)| \leq \mu < +\infty$ *for any* $t \in \mathbb{R}$ *and* $x \in \mathbb{R}^n$.
- **(H2)** $|f(t,x_1) - f(t,x_2)| \leq \gamma |x_1 - x_2|$ *for any* $t \in \mathbb{R}$, *where* $|\cdot|$ *denotes a norm in* $\mathbb{R}^n$.
- **(H3)** *There exist some constants* $K \geq 1$ *and* $0 < \alpha < +\infty$ *such that the transition matrix* $\Psi(t,s) = \Psi(t)\Psi^{-1}(s)$ *of* (1.1) *verifies*

$$||\Psi(t,s)|| \leq K e^{-\alpha(t-s)}, \quad \text{for any} \quad t \geq s. \tag{1.3}$$

- **(H4)** *The Lipschitz constant of* $f$ *verifies:*

$$\gamma \leq \alpha/4K, \tag{1.4}$$

*are satisfied, there exists a unique function* $H\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ *such that*

i) $H(t,x) - x$ *is bounded in* $\mathbb{R} \times \mathbb{R}^n$*;*
ii) *If* $t \mapsto x(t)$ *is a solution of* (1.2), *then* $H[t,x(t)]$ *is a solution of* (1.1).