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An Evans-function approach to spectral stability of internal solitary waves in stratified fluids

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Abstract

Frequently encountered in nature, internal solitary waves in stratified fluids have been investigated experimentally, theoretically, and numerically. Mathematically, these waves are exact solutions of the incompressible 2D Euler equations. Contrasting with a rich existence theory and the development of methods for their computation, their stability analysis has hardly received attention at a rigorous mathematical level.

This paper proposes a new approach to the investigation of stability of internal solitary waves in a continuously stratified fluid and carries out the following four steps of this approach: (I) to formulate the eigenvalue problem as an infinite-dimensional spatial-dynamical system, (II) to introduce finite-dimensional truncations of the spatial-dynamics description, (III) to demonstrate that each truncation, of any order, permits a well-defined Evans function, (IV) to prove absence of small zeros of the Evans function in the small-amplitude limit. The latter notably implies the low-frequency spectral stability of small-amplitude waves to arbitrarily high truncation order.

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1. Introduction

Fluidic media that are stratified according to varying density, as for example lakes, oceans, and atmospheres, typically permit the development and propagation of so-called *internal waves*,

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which, in contrast to the familiar surface waves, chiefly displace fluid elements far beneath the surface. Internal waves which are close to some quiescent state both far ahead and far astern the wave are called *internal solitary waves* (ISWs). As ISWs provide important mechanisms for mixing and energy transport and thus have direct ecological implications, the fields of oceanography, limnology, and atmosphere science have devoted considerable attention to their observation and description, see [3,13,26,45].

The "channel model" widely used in this context is given by the 2D Euler equations for incompressible fluids posed on a strip of constant finite height and infinite horizontal extent. Mathematical results based on this model broadly show the existence of solitary waves (see below), but the stability of ISWs within the channel model is an open problem. There exist numerous works on comparatively simple model equations, e.g. Korteweg–deVries equation, extended Korteweg–deVries equation, and intermediate long-wave equation, to name just a few, for which the question of stability of solitary waves has been answered comprehensively. These results certainly have implications on the stability properties of ISWs in the full-Euler channel-model setting. But, to the best of our knowledge, no stability analysis has been conducted in this setting as yet.

The central object of the present paper is a sequence of $4(N+1)$-dimensional systems, $(E_N)_{N\in\mathbb{N}}$, of ordinary differential equations on the real line that are associated with the spectral stability problem of ISWs in the channel model. We prove two main results on these "truncated eigenvalue problems", one for essentially arbitrary ISWs, the other for ISWs of small amplitude. The result for arbitrary ISWs (Theorem III in Section 3) establishes what is called consistent splitting on the closed right half $\overline{\mathbb{C}_+}$ of the complex plane, and thus the possibility of properly defining an Evans function, D_N, on $\overline{\mathbb{C}_+}$ for each E_N. The result for small-amplitude ISWs (Theorem IV in Section 4) shows that each E_N has no bounded solutions for small non-negative-real-part values of the spectral parameter, i.e., there do not exist unstable modes of small frequency.

Before obtaining these rigorous results on the truncated problems E_N, we motivate the E_N from the linearization of the full Euler equations about the ISW profile. This is done in two steps. The first step consists in showing that the eigenvalue problem derived from this linearization can be cast in a spatial-dynamics formulation, E, on $(\mathcal{L}^2(0,1))^4$ (Theorem I in Section 2). In the second step, we apply a Galerkin type procedure to E and obtain the 'truncations' E_N (Theorem II in Section 2). We emphasize that these 'derivations' of E and the E_N are completely formal; no attempts are made in the present paper to give the (certainly ill-posed!) 'infinite-dimensional dynamical system' E a rigorous interpretation, to even only formulate spectral stability at its level, or to show that the E_N approximate E in a rigorous sense.[1]

It does seem to the author, however, that Theorems III and IV would be very strange coincidences if the E_N did not, despite the formal nature of their deduction in Theorems I and II, capture essential features of the ISW stability problem in the original full-Euler channel-model setting. In particular, we consider Theorem III as a meaningful characterization of stability properties of internal solitary waves of arbitrary amplitude and Theorem IV as significant evidence for the stability of internal solitary waves of small amplitude.

In the following, we recapitulate previous work which has motivated our approach. Kirchgässner proposed what is now, generally, called "spatial dynamics" in his study [32] of an elliptic PDE[2] posed, as in our context, on a two-dimensional channel. In this approach, the unbounded

[1] But these three questions admittedly are topics of ongoing work.

[2] He did indeed consider a variant of the Dubreil–Jacotin equation that governs ISWs, see Section 1 below.