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# The Cauchy problem for a higher order shallow water type equation on the circle

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**Abstract**

In this paper, we investigate the Cauchy problem for a higher order shallow water type equation

$$u_t - u_{txx} + \partial_x^{2j+1} u - \partial_x^{2j+3} u + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0,$$

where $x \in \mathbf{T} = \mathbf{R}/2\pi$ and $j \in N^+$. Firstly, we prove that the Cauchy problem for the shallow water type equation is locally well-posed in $H^s(\mathbf{T})$ with $s \geq -\frac{j-2}{2}$ for arbitrary initial data. By using the $I$-method, we prove that the Cauchy problem for the shallow water type equation is globally well-posed in $H^s(\mathbf{T})$ with $\frac{2j+1-j^2}{2j+1} < s \leq 1$. Our results improve the results of A.A. Himonas, G. Misiolek (Himonas and Misiolek, 1998 [11], 2000 [12]).



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## 1. Introduction

In this paper, we consider the Cauchy problem for a higher order shallow water type equation

$$u_t - u_{txx} + \partial_x^{2j+1}u - \partial_x^{2j+3}u + 3uu_x - 2u_xu_{xx} - uu_{xxx} = 0, \tag{1.1}$$

$$u(x,0) = u_0(x), \quad x \in \mathbf{T} = \mathbf{R}/2\pi, \tag{1.2}$$

which is considered as the higher modification of the Camassa–Holm equation. Rewrite (1.1) as follows:

$$u_t + \partial_x^{2j+1}u + \frac{1}{2}\partial_x(u^2) + \partial_x(1-\partial_x^2)^{-1}\left[u^2 + \frac{1}{2}u_x^2\right] = 0, \tag{1.3}$$

Camassa–Holm equation corresponds to the choice $j = 0$ in (1.1), which was derived by Camassa and Holm as a nonlinear model for water wave motion in shallow channels with the aid of an asymptotic expansion directly in the Hamiltonian for Euler equations [4,8]. Compared with the KdV equation, the higher order terms arise because of the desire to go beyond the regime of waves of small amplitude, to capture waves of moderate amplitude, see [6]. Omitting the last term yields

$$u_t + \partial_x^{2j+1}u + \frac{1}{2}\partial_x(u^2) = 0. \tag{1.4}$$

When $j = 1$, equation (1.4) reduces to the Korteweg–de Vries (KdV) equation

$$u_t + u_{xxx} + \frac{1}{2}\partial_x(u^2) = 0. \tag{1.5}$$

Kenig et al. [18,19] proved that $s = -3/4$ is the critical Sobolev index for the KdV equation in real line and proved that the Cauchy problem for the periodic KdV equation is locally well-posed in $H^s(0, 2\pi\lambda)$ with $s \geq -\frac{1}{2}$ and $\lambda \geq 1$. Guo [10] and Kishimoto [20] proved that the Cauchy problem for the KdV equation is globally well-posed in $H^{-3/4}(\mathbf{R})$. Bourgain [2] proved that the Cauchy problem for the periodic KdV equation is ill-posed in $H^s(0, 2\pi\lambda)$ with $s < -\frac{1}{2}$ and $\lambda \geq 1$. Colliander et al. [5] proved that the Cauchy problem for the periodic KdV equation is globally well-posed in $H^s(0, 2\pi\lambda)$ with $s \geq -\frac{1}{2}$ and $\lambda \geq 1$. Kappeler and Topalov [14,15] proved the global well-posedness of the KdV and the defocusing mKdV equations in $H^s(0, 2\pi\lambda)$ for respectively $s \geq -1$ and $s \geq 0$ and $\lambda \geq 1$ with a solution-map which is continuous from $H^{-1}(0, 2\pi\lambda)$ $(L^2(0, 2\pi\lambda))$ into $C(R; H^{-1}(0, 2\pi\lambda))$ $(C(R; L^2(0, 2\pi\lambda)))$ with $\lambda \geq 1$. Molinet [22,24] proved that the Cauchy problem for the periodic KdV equation is ill-posed in $H^s(0, 2\pi\lambda)$ with $s < -1$ and $\lambda \geq 1$ in the sense that the solution-map associated with the KdV equation is discontinuous for the $H^s(0, 2\pi\lambda)$ topology for $s < -1$.

Lots of people have investigated the Cauchy problem for (1.3), for instance, see [3,4,8,11–13, 16,17,21,23,26–28]. In contrast to the Camassa–Holm equation, one loses integrability in the sense that the equation (1.1) is equivalent to a linear flow at constant speed in the right action-angle-variables, see the discussion in [7], but, as shown in Theorem 1.2 of the present paper, global existence prevails while for Camassa–Holm equation the development of blow-up in finite time is quite frequent. Himonas and Misiolek [11] proved that the Cauchy problem for (1.1)