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Third order differential equations describing pseudospherical surfaces

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Abstract

Third order differential equations which describe pseudospherical surfaces are considered. The complete classification of a class of such equations is given. A linear problem with one or more parameters, also known as zero curvature representation, for which the equation is the integrability condition, is explicitly given. The classification provides five large families of differential equations. Third order nonlinear dispersive wave equations, such as the Camassa–Holm equation and Degasperis–Procesi equation are examples contained in the classification. Many other explicit examples are included.

MSC: 35G20; 35Q35; 35Q51; 47J35; 53B20

Keywords: Third order differential equations; Pseudospherical surfaces; Spherical surfaces; Nonlinear partial differential equations

1. Introduction

The class of equations that describe pseudospherical surfaces contains many nonlinear partial differential equations which represent physical phenomena and that are known to be "integrable" in some broad sense. The importance of such differential equations is due to the fact that any

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equation that describes pseudospherical surfaces is the integrability condition of a linear problem with one or more parameters. This linear problem may be used to apply the inverse scattering method [14,1], where the parameter is the spectral parameter. Moreover, the linear problem may also be used to obtain Bäcklund transformations [11] or infinite number of conservation laws [9].

Since 1862, the sine-Gordon equation is known to describe surfaces with constant negative curvature. However, in the last three decades, many other well-known equations were shown to belong to this class, such as KdV [23], MKdV [10], NLS [13], Burgers [11], Camassa–Holm equation [21], etc. In fact there are huge classes of such equations. A generic solution of such an equation defines a metric on an open set of the plane, whose Gaussian curvature is $K = -1$.

A systematic study of differential equations that describe pseudospherical surfaces started with the seminal paper by Chern and Tenenblat [11]. These results were extended in a series of papers [8,13,15–20], providing many new families of differential equations. We observe that there is a similar theory for differential equations that describe spherical surfaces (metrics with positive constant curvature) [13]. However, the classification results are not as rich as those for negative curvature.

The inverse scattering method was applied to differential equations which describe pseudospherical surfaces, for example, in [4], by Beals, Rabelo and Tenenblat for equations that are not of AKNS type. Moreover, the method was also successfully applied to n-dimensional geometric generalizations of the wave and sine-Gordon equations [2,3]. These are systems of differential equations, that describe n-dimensional manifolds with constant sectional curvature (see also [25], [26] and [7]).

A differential equation for a real function $u(x, t)$ is said to describe pseudospherical surfaces if it is equivalent to the structure equations, $d\omega_1 = \omega_3 \wedge \omega_2$, $d\omega_2 = \omega_1 \wedge \omega_3$, $d\omega_3 = \omega_1 \wedge \omega_2$, of a 2-dimensional Riemannian manifold whose Gaussian curvature $K = -1$, with $\omega_i = f_{i1}dx + f_{i2}dt$, $1 \leq i \leq 3$ ($\omega_3 := \omega_{12}$ is the connection form), where f_{ij} are smooth functions of u and its derivatives.

Chern and Tenenblat [11] provided a complete characterization of the evolution equations of type $u_t = F(u, u_x, \ldots, \partial_x^k u)$ with the assumption that $f_{21} = \eta$ is a parameter. With this same assumption, Jorge, Rabelo and Tenenblat studied equations of type $u_{tt} = F(u, u_x, u_{xx}, u_t)$ in [16] and of type $u_t = u_{xxx} + G(u, u_x, u_{xx})$ in [19]. In 1989, Rabelo [18] studied differential equations of type $u_{xt} = F(u, u_x, \ldots, \partial_x^k u)$, with $2 \leq k \leq 3$. A particular example, in this class, is the now called Rabelo's cubic equation, $u_{xt} = u + (u^3)_{xx}/6$. It is interesting to observe that 25 years later, this equation was shown to describe the propagation of ultra-short light pulses in silica optical fibers [24]. Such ultra-short pulses seem to be important for future technologies of ultra-fast optical data transmission [22].

The results in [11] were generalized by Kamran and Tenenblat [17], who suppressed the *a priori* condition on f_{21}. Reyes [20] extended some aspects of the theory to the case when F depends explicitly on the variables x and t. Differential systems describing pseudospherical surfaces or spherical surfaces were studied by Ding and Tenenblat [13] in 2002.

In this paper, we are interested in studying third order differential equations of type

$$u_t - u_{xxt} = \lambda u u_{xxx} + G(u, u_x, u_{xx}), \quad \lambda \in \mathbb{R}, \tag{1.1}$$

that describe spherical or pseudospherical surfaces. Examples of such equations are the Camassa–Holm equation and the Degasperis–Procesi equation. In order to be able to treat