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Infinitely many spike solutions for the Hénon equation with critical growth ☆

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Abstract

Following the constructive method of Wei and Yan [22], with new ingredients to take care of $n = 3$, we prove the existence of infinitely many solutions of the Hénon equation $-\Delta u = |x|^{\alpha} u^{\frac{n+2}{n-2}}$ in the unit ball of $\mathbb{R}^n$ $(n \geqslant 3, \alpha > 0)$ with the Dirichlet boundary condition.
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1. Introduction

Consider the following Hénon equation: for $u = u(x)$,

$$-\Delta u = |x|^{\alpha} u^{p}, \quad u > 0 \quad \text{in } B, \qquad u = 0 \quad \text{on } \partial B, \tag{1.1}$$

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where $\alpha > 0$ is a constant, $p = (n+2)/(n-2)$ is the critical Sobolev exponent, and $B = \{x \in \mathbb{R}^n : |x| < 1\}$.

The original Hénon problem, modeling mass distribution in spherical symmetric clusters of stars [13], is introduced by Hénon in 1973. Problem (1.1) has been investigated by many researchers. In 1982, Ni [15] for the first time proved rigorously the existence of radial solutions of (1.1) for each $\alpha > 0$ and $p \in (1, \frac{n+2+2\alpha}{n-2})$. There are many works concerning the energy and the profile of the ground state (constrained energy minimizer) solutions of (1.1) for either $p \approx \frac{n+2}{n-2}$ or $\alpha \gg 1$; see, for example, [1,2,5–11,14–22], and the references therein. Smets, Su, and Willem [19,20], based on numerical discovery of Chen, Ni, and Zhou [9], proved that for every fixed $p \in (1, \frac{n+2}{n-2})$ and α large enough, or for any fixed $\alpha > 0$ and p sufficiently close to $\frac{n+2}{n-2}$ from below, any ground state solution of (1.1) is not radial. Serra [18] studied the case $p = \frac{n+2}{n-2}$ and proved the existence of non-radial positive solutions of (1.1) for $n \geqslant 4$ and α sufficiently large. Badiale and Serra [1] proved the existence of non-radial positive solutions of (1.1) for a range of p covering subcritical, critical and supercritical growth when $n \geqslant 4$ and α is sufficiently large. Cao, Peng, and Yan [7,8], as well as Byeon and Wang [5,6], proved that the points of maximum of ground states approach the boundary ∂B as $p \nearrow \frac{n+2}{n-2}$ or $\alpha \to \infty$; they also obtained limiting profile of the ground state. Multiple peak solutions for slightly subcritical growth were established by Pistoia and Serra [17] and Peng [16], and for slightly supercritical ($\alpha = 0$) by del Pino, Felmer and Musso [10]. Recently, Wei and Yan [22] proved that (1.1) admits infinitely many solutions when $n \geqslant 4$, $p = \frac{n+2}{n-2}$ and $\alpha > 0$. This paper is aimed to prove the following:

Theorem 1. *If $\alpha > 0$, $n \geqslant 3$, and $p = \frac{n+2}{n-2}$, problem (1.1) admits infinitely many solutions.*

Theorem 1 for the case $n \geqslant 4$ has been studied by Wei and Yan in [22] by using a finite dimensional reduction argument (cf. [10,14,16,17,21]) equipped with a carefully chosen weighted L^∞ norm. Here we complete the theory of Wei and Yan to cover the case $n = 3$. To our knowledge, up to now, there is no existence result of non-radial positive solution for (1.1) when $n = 3$ and $p = (n+2)/(n-2)$.

Inspired by [22], we introduce new techniques and more estimates to complement the classical method of Wei and Yan and to overcome technical difficulties arising from the case when $n = 3$. Compared with [22], our construction of approximate solutions is explicit and therefore much simpler.

The solution constructed in [22] has k peaks, with k an arbitrarily large integer. The possibility of the existence of multi-peak solutions is due to the local stability (modulo dilation and translation) of radially symmetric ground state in $\mathbb{R}^n$; see Bianchi and Egnell [4] for the Laplace operator and Bartsch, Weth, and Willem [3] for polyharmonic operators. The equation $-\Delta u = u^p$ admits a ground state $u(x) = \Phi(x) = c_n(1+|x|^2)^{-m}$ where $m = (n-2)/2$ and c_n is a positive constant. By scaling and translation, for each parameter $\varepsilon > 0$ and $\xi \in \mathbb{R}^n$,

$$u(x) = \frac{1}{\varepsilon^m}\Phi\Big(\frac{x-\xi}{\varepsilon}\Big) = \frac{c_n \varepsilon^m}{[\varepsilon^2 + |x-\xi|^2]^m}$$

is also a solution of $-\Delta u = u^p$. When $0 < \varepsilon \ll 1$, we call $\varepsilon^{-m}\Phi([x-\xi]\varepsilon^{-1})$ a spike centered at ξ. For each large enough integer k, we search for a k-spike solution of the form

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