Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 258 (2015) 4103–4126

Journal of Differential Equations

www.elsevier.com/locate/jde

The local well-posedness, existence and uniqueness of weak solutions for a model equation for shallow water waves of moderate amplitude

Shouming Zhou

College of Mathematics Science, Chongqing Normal University, Chongqing 401331, PR China

Received 1 August 2013

Available online 25 March 2015

Abstract

This paper deals with the Cauchy problem for a nonlinear equation modeling the evolution of the free surface for waves of moderate amplitude in the shallow water regime, which was proposed by Constantin and Lannes (2009) [20]. Applying the pseudoparabolic regularization technique, the local well-posedness of strong solutions in Sobolev space $H^s(\mathbb{R})$ with $s > 3/2$ is established via a limiting procedure. Moreover, a sufficient condition for the existence of weak solutions of the equation in lower order Sobolev space $H^s(\mathbb{R})$ with $1 < s \le 3/2$ is obtained.

MSC: 35G25; 35L05; 35Q50

Keywords: Quasi-linear hyperbolic equation; Local well-posedness; Weak solution; Pseudoparabolic regularization method

1. Introduction

The present paper focuses on the Cauchy problem for a model equation for shallow water waves of moderate amplitude

E-mail address: zhoushouming76@163.com.

http://dx.doi.org/10.1016/j.jde.2015.01.014

0022-0396/

$$\begin{cases} \eta_t + \eta_x + \frac{3}{2}\epsilon\eta\eta_x + \iota\epsilon^2\eta^2\eta_x + \kappa\epsilon^3\eta^3\eta_x + \mu(\alpha\eta_{xxx} + \beta\eta_{xxt}) \\ \quad = \epsilon\mu(\gamma\eta\eta_{xxx} + \delta\eta_x\eta_{xx}), & x \in \mathbb{R}, t > 0, \\ \eta(x,0) = \eta_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $\alpha, \gamma, \delta, \iota, \kappa$ and $\beta < 0$ are parameters, ϵ is amplitude parameter and μ is shallowness parameter.

If the shallowness parameter $\mu \ll 1$ and the amplitude parameter $\varepsilon = O(\mu)$, which characterized the shallow water regime of waves of small amplitude. A typical example is the Korteweg–de Vries (KdV) equation:

$$u_t - 6uu_x + u_{xxx} = 0, \tag{1.2}$$

where u describes the free surface of water (the physical derivation of this equation can be found in [1]). The beautiful structure behind the KdV equation initiated a lot of mathematical investigations. For instance, the KdV equation is completely integrable and its solitary waves are solitons (see [36]). The local and global existence of the solutions to (1.2) were proven in [38]. The well-posedness and scattering results for the generalized Korteweg–de Vries equation are studied via the contraction principle (see [32]). It is also observed that the KdV equation does not accommodate wave breaking (by wave breaking we mean that the wave remains bounded but its slope becomes unbounded in finite time, cf. [39]).

While the regime of waves of shallow water waves of moderate amplitude corresponds to shallowness parameter $\mu \ll 1$ and amplitude parameter $\varepsilon = O(\sqrt{\mu})$, one of the closest relatives model is the Camassa–Holm equation

$$u_t - u_{xxt} + c_0u_x + 3uu_x - 2u_xu_{xx} - uu_{xxx} = 0. \tag{1.3}$$

Camassa–Holm equation arises in a variety of different contexts. In 1981, it was originally derived as a bi-Hamiltonian equation with infinitely many conservation laws by Fokas and Fuchssteiner [27]. It has been widely studied since 1993 when Camassa and Holm [6] proposed it as a model for the unidirectional propagation of shallow water waves over a flat bed. Such as, Camassa–Holm equation has also a bi-Hamiltonian structure [27,34] and is completely integrable [6,10,30], and it possesses an infinity of conservation laws and is solvable by its corresponding inverse scattering transform (cf. [4,8,18]). The stability of the smooth solitons was considered in [23], and the orbital stability of the peaked solitons were proved in [22]. It is worth pointing out that solutions of this type are not mere abstractions: the peakons replicate a feature that is characteristic for the waves of great height-waves of largest amplitude that are exact solutions of the governing equations for irrotational water waves (see [15] and references therein). The explicit interaction of the peaked solitons were given in [2]. It has been shown that this problem is locally well-posed for initial data $u_0 \in H^s$ with $s > \frac{3}{2}$ (cf. [12,24,35]). Moreover, Camassa–Holm equation not only has global strong solutions, but also admits finite time blow-up solutions [9,12,13,19,35], and the blow-up occurs in the form of breaking waves, namely, the solution remains bounded but its slope becomes unbounded in finite time. On the other hand, it also has global weak solutions in H^1 (see [5,14,21,41]). The advantage of the Camassa–Holm equation in comparison with the KdV equation lies in the fact that the Camassa–Holm equation has peaked solitons and models the peculiar wave breaking phenomena (cf. [7,13]).

Since quantities of order $\varepsilon = O(\sqrt{\mu})$ are also of order $O(\mu)$ for $\mu \ll 1$, the regime of moderate amplitude captures a wider range of wave profiles. In particular, within this regime one