Available online at www.sciencedirect.com
ScienceDirect
J. Differential Equations 258 (2015) 4156–4179
Journal of Differential Equations
www.elsevier.com/locate/jde

Semiclassical analysis for pseudo-relativistic Hartree equations

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Received 22 September 2014; revised 13 January 2015
Available online 9 February 2015

Abstract

In this paper we study the semiclassical limit for the pseudo-relativistic Hartree equation

$$\sqrt{-\varepsilon^2\Delta + m^2}u + Vu = \left(I_\alpha * |u|^p\right)|u|^{p-2}u, \quad \text{in } \mathbb{R}^N,$$

where $m > 0$, $2 \le p < \frac{2N}{N-1}$, $V\colon \mathbb{R}^N \to \mathbb{R}$ is an external scalar potential, $I_\alpha(x) = \frac{c_{N,\alpha}}{|x|^{N-\alpha}}$ is a convolution kernel, $c_{N,\alpha}$ is a positive constant and $(N-1)p - N < \alpha < N$. For $N = 3$, $\alpha = p = 2$, our equation becomes the pseudo-relativistic Hartree equation with Coulomb kernel.

Keywords: Pseudo-relativistic Hartree equations; Semiclassical limit

1. Introduction

In this paper we study the semiclassical limit ($\varepsilon \to 0^+$) for the pseudo-relativistic Hartree equation

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http://dx.doi.org/10.1016/j.jde.2015.01.029
0022-0396/

$$i\varepsilon\frac{\partial\psi}{\partial t} = \big(\sqrt{-\varepsilon^2\Delta + m^2} - m\big)\psi + V\psi - \left(\frac{1}{|x|} * |\psi|^2\right)\psi, \quad x \in \mathbb{R}^3 \tag{1}$$

where $\psi: \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}$ is the wave field, $m > 0$ is a physical constant, ε is the semiclassical parameter $0 < \varepsilon \ll 1$, a dimensionless scaled Planck constant (all other physical constants are rescaled to be 1), V is bounded external potential in $\mathbb{R}^3$. Here the pseudo-differential operator $\sqrt{-\varepsilon^2\Delta + m^2}$ is simply defined in Fourier variables by the symbol $\sqrt{\varepsilon^2|\xi|^2 + m^2}$ (see [23]).

Eq. (1) has interesting applications in the quantum theory for large systems of self-interacting, relativistic bosons with mass $m > 0$. As recently shown by Elgart and Schlein [16], Eq. (1) emerges as the correct evolution equation for the mean-field dynamics of many-body quantum systems modelling pseudo-relativistic boson stars in astrophysics. The external potential, $V = V(x)$, accounts for gravitational fields from other stars. In what follows, we will assume that V is a smooth, bounded function (see [24,19,17,18,21,28]). The pseudo-relativistic Hartree equation can be also derived coupling together a pseudo-relativistic Schrödinger equation with a Poisson equation (see for instance [1,32]), i.e.

$$\begin{cases} i\varepsilon\dfrac{\partial\psi}{\partial t} = \big(\sqrt{-\varepsilon^2\Delta + m^2} - m\big)\psi + V\psi - U\psi, \\ -\Delta U = |\psi|^2. \end{cases}$$

See also [14,20] for recent developments for models involving pseudo-relativistic Bose gases.

Solitary wave solutions $\psi(t,x) = e^{it\lambda/\varepsilon}u(x)$, $\lambda > 0$ to Eq. (1) lead to solve the non-local single equation

$$\sqrt{-\varepsilon^2\Delta + m^2}\,u + Vu = \left(\frac{1}{|x|} * |u|^2\right)u, \quad \text{in } \mathbb{R}^3 \tag{2}$$

where for simplicity we write V instead of $V + (\lambda - m)$.

More generally, in this paper we will study the generalized pseudo-relativistic Hartree equation

$$\sqrt{-\varepsilon^2\Delta + m^2}\,u + Vu = \big(I_\alpha * |u|^p\big)|u|^{p-2}u, \quad \text{in } \mathbb{R}^N, \tag{3}$$

where $m > 0$, $2 \le p < \frac{2N}{N-1}$, $V: \mathbb{R}^N \to \mathbb{R}$ is an external scalar potential,

$$I_\alpha(x) = \frac{c_{N,\alpha}}{|x|^{N-\alpha}} \quad (x \neq 0),\ \alpha \in (0, N)$$

is a convolution kernel and $c_{N,\alpha}$ is a positive constant; for our purposes we can choose $c_{N,\alpha} = 1$. For $N = 3$, $\alpha = p = 2$, Eq. (3) becomes the pseudo-relativistic Hartree equation (2) with Coulomb kernel.

We refer to [34,9,6,30] for the semiclassical analysis of the non-relativistic Hartree equation. The study of the pseudo-relativistic Hartree equation (2) without external potential V starts in the pioneering paper [24] where Lieb and Yau, by minimization on the sphere $\{\phi \in L^2(\mathbb{R}^3) \mid \int_{\mathbb{R}^3}|\phi|^2 = M\}$, proved that a radially symmetric ground state exists in $H^{1/2}(\mathbb{R}^3)$ whenever $M < M_c$, the so-called Chandrasekhar mass. Later Lenzmann proved in [22] that this ground state is unique (up to translations and phase change) provided that the mass M is sufficiently small; some results about the non-degeneracy of the ground state solution are also given.