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# Wave equation for sums of squares on compact Lie groups

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**Abstract**

In this paper we investigate the well-posedness of the Cauchy problem for the wave equation for sums of squares of vector fields on compact Lie groups. We obtain the loss of regularity for solutions to the Cauchy problem in local Sobolev spaces depending on the order to which the Hörmander condition is satisfied, but no loss in globally defined spaces. We also establish Gevrey well-posedness for equations with irregular coefficients and/or multiple characteristics. As in the Sobolev spaces, if formulated in local coordinates, we observe well-posedness with the loss of local Gevrey order depending on the order to which the Hörmander condition is satisfied.



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*Keywords:* Wave equation; Sub-Laplacian; Sum of squares; Well-posedness; Sobolev spaces; Gevrey spaces

## 1. Introduction

In this paper we investigate the well-posedness of the Cauchy problem for time-dependent wave equations associated to sums of squares of invariant vector fields on compact Lie groups.

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Such analysis is motivated, in particular, by general investigations of the well-posedness and wave propagation governed by subelliptic operators and problems with multiplicities. An often encountered example of subelliptic behaviour is a sum of squares of vector fields, extensively analysed by Hörmander [17,18], Oleinik and Radkevich [27], Rothschild and Stein [28], and by many others. For invariant operators on compact Lie groups, the sum of squares becomes formally self-adjoint, making the corresponding wave equation hyperbolic, a necessary condition for the analysis of the corresponding Cauchy problem. Already in this setting, we discover a new phenomenon of the loss of the local Gevrey regularity for its solutions. Moreover, this loss is linked to the order to which the Hörmander condition is satisfied.

Thus, let $G$ be a compact Lie group of dimension $n$ with Lie algebra $\mathfrak{g}$, and let $X_1, \ldots, X_k$ be a family of left-invariant vector fields in $\mathfrak{g}$. Let

$$\mathcal{L} := X_1^2 + \cdots + X_k^2 \tag{1.1}$$

be the sum of squares of derivatives defined by the vector fields. If the iterated commutators of $X_1, \ldots, X_k$ span the Lie algebra of $G$, the operator $\mathcal{L}$ is a sub-Laplacian on $G$, hypoelliptic in view of Hörmander's sum of the squares theorem.

With or without the Hörmander condition, it can be shown that the operator $\partial_t^2 - \mathcal{L}$ is (weakly) hyperbolic (see Remark 3.2). For a continuous function $a = a(t) \geq 0$, we will be concerned with the Cauchy problem

$$\begin{cases} \partial_t^2 u(t,x) - a(t)\mathcal{L}u(t,x) = 0, & (t,x) \in [0,T] \times G, \\ u(0,x) = u_0(x), & x \in G, \\ \partial_t u(0,x) = u_1(x), & x \in G. \end{cases} \tag{1.2}$$

When localised, the Cauchy problem (1.2) is a weakly hyperbolic equation with both time and space dependent coefficients, and the available results and techniques are rather limited compared to, for example, the situation when the coefficients depend only on time. For example, general Gevrey well-posedness results of Bronshtein [3] or Nishitani [26] may apply for some $a$ and $\mathcal{L}$, but in general they do not take into account the geometry of the problem and of the operator $\mathcal{L}$.

In the case of the Euclidean space $\mathbb{R}^n$, the Cauchy problem for the operator $\partial_t^2 - a(t)\Delta$ with the Laplacian $\Delta$ has been extensively studied. It is known that the Cauchy problem for this operator may be not well-posed in $C^\infty(\mathbb{R}^n)$ and in $\mathcal{D}'(\mathbb{R}^n)$ if the function $a(t)$ becomes zero or is irregular, see, respectively, Colombini and Spagnolo [7], and Colombini, Jannelli and Spagnolo [6]. Thus, Gevrey spaces appear naturally in such well-posedness problems already on $\mathbb{R}^n$, and for the latter equation, a number of sharp well-posedness results in Gevrey spaces have been established by Colombini, de Giorgi and Spagnolo [5]. We note that problems with lower (e.g. distributional) regularity of coefficients require different methods, see e.g. the authors' paper [14]. At the same time, even for analytic principal part, inclusion of lower order terms may require suitable Levi conditions, see e.g. [13].

Our analysis will cover the case of the Laplacian $\Delta$ on the compact Lie group $G$ since we can write it as $\mathcal{L} = X_1^2 + \cdots + X_n^2$ for a basis $X_1, \ldots, X_n$ of the Lie algebra of $G$. For different ways of representing Laplacians on compact Lie groups we refer to an extensive discussion in Stein [33]. In the case of the Laplacian we recover the orders that can be obtained from the work of Nishitani [26] since in this case we can write $\mathcal{L}$ in local coordinates in the divergence form. For sub-Laplacian $\mathcal{L}$ this no longer applies (neither are the results of Jannelli [19] because of the lack of divergence form and appearing lower order terms).