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Divergence and Poincaré–Liapunov constants for analytic differential systems

Maite Grau [a,*], Jaume Llibre [b]

[a] Departament de Matemàtica, Universitat de Lleida, Avda. Jaume II, 69, 25001 Lleida, Catalonia, Spain
[b] Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

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Abstract

We consider a planar autonomous real analytic differential system with a monodromic singular point p. We deal with the center problem for the singular point p. Our aim is to highlight some relations between the divergence of the system and the Poincaré–Liapunov constants of p when these are defined.

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1. Introduction and statement of the main results

Let O be the origin of coordinates of $\mathbb{R}^2$ and let $\mathcal{U}_O$ be a neighborhood of O. We consider two real analytic functions $P(x, y)$ and $Q(x, y)$ in $\mathcal{U}_O$ which vanish at O. In this work we deal with the analytic differential systems of the form

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y), \tag{1}$$

where the dot denotes derivative with respect to an independent real variable t.

* Corresponding author.
E-mail addresses: mtgrau@matematica.udl.cat (M. Grau), jllibre@mat.uab.cat (J. Llibre).

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When all the orbits of system (1) in a punctured neighborhood of the singular point O are periodic, we say that the origin is a *center*. If the orbits of system (1) in a punctured neighborhood of O are spiral to O when $t \to +\infty$ or $t \to -\infty$, then we say that the origin is a *focus*. In the first case ($t \to +\infty$), we say that it is *stable* and in the second case ($t \to -\infty$), we say that it is *unstable*. If the origin is either a focus or a center, we say that it is a *monodromic* singular point. The *center problem* consists in distinguishing when a monodromic singular point is either a center or a focus. *In the sequel we assume that the origin of system (1) is monodromic.*

As usual we define the *divergence of system (1)*, and we denote it by $\operatorname{div}(x, y)$, as the function

$$\operatorname{div}(x, y) = \frac{\partial P}{\partial x}(x, y) + \frac{\partial Q}{\partial y}(x, y).$$

System (1) is said to be *Hamiltonian* if $\operatorname{div}(x, y) \equiv 0$. In such a case there exists a neighborhood of the origin $\mathcal{U}_O$ and an analytic function $H : \mathcal{U}_O \subseteq \mathbb{R}^2 \to \mathbb{R}$, called the Hamiltonian, such that

$$P(x, y) = -\frac{\partial H}{\partial y} \quad \text{and} \quad Q(x, y) = \frac{\partial H}{\partial y}.$$

We note that the level curves of H are formed by orbits of system (1). A Hamiltonian system (1) with a monodromic singular point at O necessarily has a center at the origin because an analytic function cannot contain a spiral as level curve (unless the analytic function be constant).

Our aim is to highlight some other results relating the divergence of system (1) with the solution of the center problem.

Given a real analytic function $f : \mathcal{U}_O \subseteq \mathbb{R}^2 \to \mathbb{R}$, where $\mathcal{U}_O$ is a neighborhood of the origin $O = (0, 0)$, we consider its Taylor expansion at O:

$$f(x, y) = f_d(x, y) + \mathcal{O}_{d+1}(x, y),$$

where $d \geq 0$ is an integer and $f_d(x, y)$ is a non-zero homogeneous polynomial of degree d. We say that f is of *sign definite* if $f_d(x, y) \geq 0$ or $f_d(x, y) \leq 0$ for all $(x, y) \in \mathbb{R}^2$. When $f_d(x, y) \geq 0$ (resp. $f_d(x, y) \leq 0$) for all $(x, y) \in \mathbb{R}^2$ we say that f is *positive definite* (resp. *negative definite*). It is clear that a necessary condition for $f(x, y)$ to be of sign definite is that d is even.

Our first result is the following one.

Proposition 1. *Assume that the origin of an analytic differential system (1) is a monodromic singular point. If the divergence* $\operatorname{div}(x, y)$ *of system (1) is of sign definite, then the origin of system (1) is a focus; either unstable if the divergence is positive definite or stable if it is negative definite.*

This result is proved in Section 2. We remark that in the case that the origin of system (1) is a strong focus, then the divergence $\operatorname{div}(0, 0) \neq 0$ and the stability of the focus is given by the sign of the number $\operatorname{div}(0, 0)$. The previous proposition is a generalization of this fact for any monodromic singular point. See for instance Theorem 2.15 of [6], or [8], for the definitions of these classical concepts.

Assume that the origin of system (1) is a monodromic singular point, but not a strong focus. It is well-known that, after a linear change of variables and a constant scaling of the time variable (if necessary), the system can be written in one of the following three forms: