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Global dynamics above the ground state energy for the focusing nonlinear Klein–Gordon equation

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A B S T R A C T

The analysis of global dynamics of nonlinear dispersive equations has a long history starting from small solutions. In this paper we study the focusing, cubic, nonlinear Klein–Gordon equation in $\mathbb{R}^3$ with large radial data in the energy space. This equation admits a unique positive stationary solution Q, called the ground state. In 1975 Payne and Sattinger showed that solutions $u(t)$ with energy $E[u,\dot{u}]$ *strictly below* that of the ground state are divided into two classes, depending on a suitable functional $K(u)$: If $K(u) < 0$, then one has finite time blow-up, if $K(u) \geqslant 0$ global existence; moreover, these sets are invariant under the flow. Recently, Ibrahim, Masmoudi and the first author [22] improved this result by establishing scattering to zero for $K[u] \geqslant 0$ by means of a variant of the Kenig–Merle method (Kenig and Merle, 2006, 2008 [25,26]). In this paper we go slightly beyond the ground state energy and we give a complete description of the evolution in that case. For example, in a small neighborhood of Q one encounters the following trichotomy: On one side of a center-stable manifold one has finite time blow-up for $t \geqslant 0$, on the other side scattering to zero, and on the manifold itself one has scattering to Q, both as $t \to +\infty$. In total, the class of data with energy at most slightly above that of Q is divided into nine disjoint non-empty sets each displaying different asymptotic behavior as $t \to \pm\infty$, which includes solutions blowing up in one time direction and scattering to zero on the other. The analogue of the solutions found by Duyckaerts and Merle (2009, 2008) [13,14] for the energy critical wave and Schrödinger equations appear here as the unique one-dimensional stable/unstable manifolds approaching $\pm Q$ exponentially as $t \to \infty$ or $t \to -\infty$, respectively. The main technical ingredient in our

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proof is a "one-pass" theorem which excludes the existence of (almost) homoclinic orbits between Q (as well as $-Q$) and (almost) heteroclinic orbits connecting Q with $-Q$. In a companion paper (Nakanishi and Schlag, 2010 [31]) we establish analogous properties for the NLS equation.

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1. Introduction

In this paper we study the global behavior of general solutions to the nonlinear Klein–Gordon equation (NLKG) with the focusing cubic nonlinearity on $\mathbb{R}^3$, i.e.,

$$\ddot{u} - \Delta u + u = u^3, \qquad u(t,x) : \mathbb{R}^{1+3} \to \mathbb{R}, \tag{1.1}$$

which conserves the energy

$$E(\vec{u}) := \int_{\mathbb{R}^3} \left[\frac{|\dot{u}|^2 + |\nabla u|^2 + |u|^2}{2} - \frac{|u|^4}{4} \right] dx. \tag{1.2}$$

We regard $H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ as the phase space for this infinite dimensional Hamiltonian system. In other words, we write the solutions as

$$\vec{u}(t) := \big(u(t), \dot{u}(t)\big) \in \mathcal{H} := H^1_{\mathrm{rad}} \times L^2_{\mathrm{rad}}. \tag{1.3}$$

There exists a vast literature on the wellposedness theory for this equation in the energy space since Jörgens [23] and Segal [35], as well as the scattering theory for small data for the focusing nonlinearity as in (1.1) and for large data for the defocusing equation; see Brenner [7,8], Ginibre and Velo [16,17], Morawetz and Strauss [30], and Pecher [33]. In this paper, the scattering of a solution u to a static state φ refers to the following asymptotic behavior: There exists a solution v of the free Klein–Gordon equation such that