J. Differential Equations 250 (2011) 2334–2346

Contents lists available at ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

The asymptotic behaviour of the heat equation in a twisted Dirichlet–Neumann waveguide

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ARTICLE INFO

Article history:
Received 10 June 2010
Available online 26 November 2010

MSC:
primary 35K05
secondary 35P20

Keywords:
Laplacian
Dirichlet and Neumann boundary conditions
Twist
Hardy inequality
Subcriticality
Heat equation
Large-time behaviour of solutions
Waveguide

ABSTRACT

We consider the heat equation in a straight strip, subject to a combination of Dirichlet and Neumann boundary conditions. We show that a switch of the respective boundary conditions leads to an improvement of the decay rate of the heat semigroup of the order of $t^{-1/2}$. The proof employs similarity variables that lead to a non-autonomous parabolic equation in a thin strip contracting to the real line, that can be analysed on weighted Sobolev spaces in which the operators under consideration have discrete spectra. A careful analysis of its asymptotic behaviour shows that an added Dirichlet boundary condition emerges asymptotically at the switching point, breaking the real line in two half-lines, which leads asymptotically to the $1/2$ gain on the spectral lower bound, and the $t^{-1/2}$ gain on the decay rate in the original physical variables.
This result is an adaptation to the case of strips with twisted boundary conditions of previous results by the authors on geometrically twisted Dirichlet tubes.

1. Introduction

We consider the heat equation

$$u_t - \Delta u = 0 \tag{1}$$

in an infinite planar strip $\Omega := \mathbb{R} \times (-a, a)$ of half-width $a > 0$, subject to

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0022-0396/$ – see front matter
doi:10.1016/j.jde.2010.11.005

Fig. 1. Planar strips with untwisted (left) and twisted (right) boundary conditions; the thick and thin lines correspond to Dirichlet and Neumann boundary conditions, respectively.

$$\begin{cases} \text{Dirichlet boundary conditions on } \Gamma_\pi^D := (-\infty, 0) \times \{-a\} \cup (0, +\infty) \times \{a\}, \\ \text{Neumann boundary conditions on } \Gamma_\pi^N := (0, +\infty) \times \{-a\} \cup (-\infty, 0) \times \{a\}, \end{cases}$$

and to the initial condition

$$u(\cdot, 0) = u_0 \in L^2(\Omega). \tag{2}$$

This model is considered as a 'twisted' counterpart of the explicitly solvable problem given by (see Fig. 1):

$$\begin{cases} \text{Dirichlet boundary conditions on } \Gamma_0^D := (-\infty, +\infty) \times \{-a\}, \\ \text{Neumann boundary conditions on } \Gamma_0^N := (-\infty, +\infty) \times \{a\}. \end{cases}$$

Henceforth we shall use the common subscript

$$\theta \in \{0, \pi\}$$

when we want to deal with the two problems simultaneously (the value of θ suggests the rotation angle giving rise to twisting/untwisting).

The solution to (1)–(2) is given by $u(t) = e^{\Delta_\theta t} u_0$, where $e^{\Delta_\theta t}$ is the semigroup operator on $L^2(\Omega)$ associated with the Laplacian $-\Delta_\theta$ determined by the respective boundary conditions (depending on θ).

The operators $-\Delta_\pi$ and $-\Delta_0$ have the same spectrum

$$\sigma(-\Delta_\theta) = \sigma_{\mathrm{ess}}(-\Delta_\theta) = [E_1, \infty), \quad \text{where } E_1 := \left(\frac{\pi}{4a}\right)^2. \tag{3}$$

Consequently, for all $t \geqslant 0$,

$$\left\| e^{\Delta_\theta t} \right\|_{L^2(\Omega) \to L^2(\Omega)} = e^{-E_1 t}, \tag{4}$$

irrespectively of the value of θ.

In this paper, we are interested in additional time decay properties of the heat semigroup, when the initial data are restricted to a subspace of the Hilbert space $L^2(\Omega)$. We restrict ourselves to the weighted space

$$L^2(\Omega, K) \quad \text{with } K(x) := e^{x_1^2/4}, \tag{5}$$

which means that the initial data are required to be sufficiently rapidly decaying at the infinity of the strip. As a measure of the additional decay, we consider the (polynomial) *decay rate*

$$\gamma_\theta := \sup\left\{ \gamma \;\middle|\; \exists C_\gamma > 0,\ \forall t \geqslant 0,\ \left\| e^{(\Delta_\theta + E_1)t} \right\|_{L^2(\Omega, K) \to L^2(\Omega)} \leqslant C_\gamma (1 + t)^{-\gamma} \right\}. \tag{6}$$

Our main result reads as follows: