J. Differential Equations 250 (2011) 4085–4103

Contents lists available at ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Stability of periodic solutions of state-dependent delay-differential equations

John Mallet-Paret [a,*,1], Roger D. Nussbaum [b,2]

[a] Division of Applied Mathematics, Brown University, Providence, RI 02912, United States
[b] Department of Mathematics, Rutgers University, Piscataway, NJ 08854, United States

ARTICLE INFO

Article history:
Received 11 October 2010
Available online 22 February 2011

MSC:
primary 34K13, 34K20, 37L15
secondary 37C27, 37C75

Keywords:
Delay-differential equations
State-dependent delay
Periodic solution
Orbital stability
Asymptotic phase

ABSTRACT

We consider a class of autonomous delay-differential equations

$$\dot{z}(t) = f(z_t)$$

which includes equations of the form

$$\begin{aligned} &\dot{z}(t) = g\big(z(t), z(t-r_1), \ldots, z(t-r_n)\big), \\ &r_i = r_i\big(z(t)\big) \quad \text{for } 1 \leqslant i \leqslant n, \end{aligned} \tag{$*$}$$

with state-dependent delays $r_i(z(t)) \geqslant 0$. The functions g and r_i satisfy appropriate smoothness conditions.

We assume there exists a periodic solution $z = x(t)$ which is linearly asymptotically stable, namely with all nontrivial characteristic multipliers μ satisfying $|\mu| < 1$. We prove that the appropriate nonlinear stability properties hold for $x(t)$, namely, that this solution is asymptotically orbitally stable with asymptotic phase, and enjoys an exponential rate of attraction given in terms of the leading nontrivial characteristic multiplier.

A principal difficulty which distinguishes the analysis of equations such as $(*)$ from ones with constant delays, is that even with g and r_i smooth, the associated function f is not smooth in function space. Techniques of Hartung, Krisztin, Walther, and Wu are employed to resolve these issues.

* Corresponding author.
E-mail addresses: jmp@dam.brown.edu (J. Mallet-Paret), nussbaum@math.rutgers.edu (R.D. Nussbaum).

[1] Partially supported by NSF DMS-0500674, and by the Center for Nonlinear Analysis, Rutgers University.
[2] Partially supported by NSF DMS-0701171, and by the Lefschetz Center for Dynamical Systems, Brown University.

0022-0396/$ – see front matter © 2010 Published by Elsevier Inc.
doi:10.1016/j.jde.2010.10.023

1. Introduction

In this paper we study stability questions for a broad class of autonomous state-dependent delay-differential equations. Specifically, we prove that linearized asymptotic stability of a periodic solution $x(t)$ implies nonlinear (Lyapunov) stability of that solution, in fact, asymptotic orbital stability with asymptotic phase, and exponential attraction at a rate determined by the leading nontrivial characteristic multiplier. This is, of course, the analog of a classic theorem in ordinary differential equations; see, for example, [1]. The corresponding result for retarded equations with constant delay also has been known for many years; see [7].

Among the equations we treat are those with pointwise state-dependent delays such as

$$\dot{z}(t) = g\big(z(t), z(t-r_1), \ldots, z(t-r_n)\big), \qquad r_i = r_i\big(z(t)\big) \quad \text{for } 1 \leqslant i \leqslant n, \tag{1.1}$$

where

$$g : U_g \subseteq \mathbf{R}^{m(n+1)} \to \mathbf{R}^m, \qquad r_i : U_{r_i} \subseteq \mathbf{R}^m \to [0, R] \quad \text{for } 1 \leqslant i \leqslant n,$$

for some (typically open) sets U_g and U_{r_i}. In the case $n = 1$ this equation takes the form

$$\dot{z}(t) = g\big(z(t), z(t-r)\big), \qquad r = r\big(z(t)\big), \tag{1.2}$$

where

$$g : U_g \subseteq \mathbf{R}^{2m} \to \mathbf{R}^m, \qquad r : U_r \subseteq \mathbf{R}^m \to [0, R]. \tag{1.3}$$

The model equation

$$\varepsilon\dot{z}(t) = -z(t) - kz(t-r), \qquad r = r\big(z(t)\big) = 1 + z(t) \tag{1.4}$$

with $\varepsilon > 0$ and $k > 1$, considered in [5] (see also [4]), is a special case.

Generally, we follow the setting of Walther [8] for state-dependent equations (see also Hartung, Krisztin, Walther, and Wu [3]), which we now outline. Consider an autonomous equation

$$\dot{z}(t) = f(z_t) \tag{1.5}$$

where

$$\begin{aligned} &f : U_X \subseteq X \to \mathbf{R}^m \quad \text{is continuous}, \qquad X = C\big([-R, 0], \mathbf{R}^m\big), \\ &z_t \in X \quad \text{is given by } z_t(\theta) = z(t+\theta) \quad \text{for } \theta \in [-R, 0], \end{aligned} \tag{1.6}$$

and where U_X is an open subset of X. This is the classic setting of Hale, as described in the book of Hale and Verduyn Lunel [2]. Local existence of the initial value problem

$$z_0 = \varphi \tag{1.7}$$

for any $\varphi \in U_X$ in forward time is guaranteed, that is, the problem (1.5), (1.7) has a solution $z(t)$ for $0 \leqslant t < \delta$ for some $\delta > 0$. Note that Eq. (1.2) falls into this class by taking

$$f(\varphi) = g\big(\varphi(0), \varphi\big(-r\big(\varphi(0)\big)\big)\big). \tag{1.8}$$