J. Differential Equations 250 (2011) 3398–3417

Contents lists available at ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Nontrivial solutions for critical potential elliptic systems

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ARTICLE INFO

Article history:
Received 2 August 2010
Revised 15 December 2010
Available online 14 January 2011

MSC:
35J65
49K20
35J40

Keywords:
Critical Sobolev exponents
Potential systems
Nonlinear eigenvalues
Coercivity

ABSTRACT

We consider potential elliptic systems involving p-Laplace operators, critical nonlinearities and lower-order perturbations. Suitable necessary and sufficient conditions for existence of nontrivial solutions are presented. In particular, a number of results on Brezis–Nirenberg type problems are extended in a unified framework.

1. Introduction and main results

In the 80s decade, Brezis and Nirenberg investigated, in the celebrated paper [6], the existence of a nontrivial solution u for the critical problem

$$\begin{cases} -\Delta u = |u|^{\frac{4}{n-2}} u + \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

on a bounded domain with smooth boundary $\Omega \subset \mathbb{R}^n$, $n \geqslant 3$, where λ is a real parameter. Since then a lot of attention has been devoted to various questions and extensions related to (1). We refer for instance to Chapter 3 of the Struwe's book [25] and references therein for an overview on the so called Brezis–Nirenberg problem.

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0022-0396
doi:10.1016/j.jde.2011.01.001

A possible extension has been addressed to the quasilinear problem

$$\begin{cases} -\Delta_p u = |u|^{p^*-2}u + \lambda |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ stands for the p-Laplace operator, $1 < p < n$ and $p^* = np/(n-p)$ denotes the critical Sobolev exponent for the embedding of $W_0^{1,p}(\Omega)$ into $L^q(\Omega)$. As is well known, the existence of a positive solution u for (2) relies strongly on the value of the parameter λ compared with the first eigenvalue λ_1 corresponding to the p-Laplace operator on Ω under Dirichlet boundary condition. In particular, let us recall that when $p = 2$, according to the main result of [6], problem (1) admits a positive solution u if, and only if, $\lambda \in]0, \lambda_1[$, provided that $n \geqslant 4$. The result has been extended by Egnell [14], Garcia Azorero and Peral Alonso [15] and Guedda and Veron [19], who have proved that problem (2) admits a positive solution u if, and only if, $\lambda \in]0, \lambda_1[$, provided that $p > 1$ and $n \geqslant p^2$. When $\lambda \in]0, \lambda_1[$, such a solution u can for instance be obtained as a least energy solution through the minimization of the functional

$$\phi(u) = \int_\Omega |\nabla u|^p \, dx + \lambda \int_\Omega |u|^p \, dx$$

constrained to the Nehari manifold $\{u \in W_0^{1,p}(\Omega): \int_\Omega |u|^{p^*} dx = 1\}$ and the fact that $|u|$ also minimizes ϕ. Moreover, it is well known that, if Ω is star-shaped, then problem (2) has no nontrivial solution u for any $\lambda < 0$, see [6,14,19]. However, when $\lambda \geqslant \lambda_1$, some results on existence of a nontrivial solution u have been established in the literature, see [7,16] for $p = 2$ and [2] and the recent work [13] for $p \neq 2$.

Another possible extension concerns the semilinear system

$$\begin{cases} -\Delta u = f(u) + g(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3}$$

where $u = (u_1, \ldots, u_k)$, $\Delta u = (\Delta u_1, \ldots, \Delta u_k)$, $k \geqslant 1$, $f(u) = \frac{1}{2^*}\nabla F(u)$ and $g(u) = \frac{1}{2}\nabla G(u)$, where $F, G : \mathbb{R}^k \to \mathbb{R}$ are C^1 functions with F positively homogeneous of degree 2^* and G homogeneous of degree 2. We recall that a function $H : \mathbb{R}^k \to \mathbb{R}$ is homogeneous of degree q when $H(\rho t) = \rho^q H(t)$ for any $\rho > 0$ and $t \in \mathbb{R}^k$. Problem (1) corresponds exactly to the choice $k = 1$, $F(t) = |t|^{2^*}$ and $G(t) = \lambda t^2$. More generally, problems (2) and (3) can be simultaneously recovered from the quasilinear system

$$\begin{cases} -\Delta_p u = f(u) + g(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{4}$$

where $u = (u_1, \ldots, u_k)$, $\Delta_p u = (\Delta_p u_1, \ldots, \Delta_p u_k)$, $f(u) = \frac{1}{p^*}\nabla F(u)$ and $g(u) = \frac{1}{p}\nabla G(u)$, where $F, G : \mathbb{R}^k \to \mathbb{R}$ are C^1 functions with F positively homogeneous of degree p^* and G homogeneous of degree p. Such systems are known in the literature as potential systems (or gradient systems) and F and G are called potential functions. When $k = 1$, problem (4) takes the form (2), since F and G become $F(t) = |t|^{p^*}$ and $G(t) = \lambda |t|^p$, modulo constant factors. For $k \geqslant 2$, there are many homogeneous potential functions. Some typical examples are:

(i) $F(t) = |t|_q^{p^*}$, $F(t) = (\pi_l(t))^{p^*/l}$;
(ii) $G(t) = |t|_q^p$, $G(t) = (\pi_l(t))^{p/l}$, $G(t) = |\langle At, t\rangle|^{(p-2)/2}\langle At, t\rangle$,

where $|t|_q := (\sum_{i=1}^n |t_i|^q)^{1/q}$ is the Euclidean q-norm with $q \geqslant 1$, π_l is the lth elementary symmetric polynomial, $l = 1, \ldots, k$, $\langle \cdot, \cdot \rangle$ denotes the usual Euclidean inner product and $A = (a_{ij})$ is a real