J. Differential Equations 250 (2011) 3150–3195

Contents lists available at ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

New estimates for the div-curl-grad operators and elliptic problems with L^1-data in the whole space and in the half-space

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ARTICLE INFO

Article history:
Received 27 July 2010
Revised 6 January 2011
Available online 5 February 2011

MSC:
35J25
35J47

Keywords:
Div-curl-grad operators
Elliptic
Half-space
Weighted Sobolev spaces

ABSTRACT

In this paper, we study the div-curl-grad operators and some elliptic problems in the whole space $\mathbb{R}^n$ and in the half-space $\mathbb{R}^n_+$, with $n \geqslant 2$. We consider data in weighted Sobolev spaces and in L^1.

1. Introduction

The purpose of this paper is to present new results concerning the div-curl-grad operators and some elliptic problems in the whole space and in the half-space with data and solutions which live in L^1 or in weighted Sobolev spaces, expressing at the same time their regularity and their behavior at infinity. Recently, new estimates for L^1-vector field have been discovered by Bourgain, Brézis and Van Schaftingen (see [22,10–13,15]) which yield in particular improved estimates for the solutions of elliptic systems in $\mathbb{R}^n$ or in a bounded domain $\Omega \subset \mathbb{R}^n$. Our work presented in this paper is naturally based on these very interesting results and our approach rests on the use of weighted Sobolev spaces.

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0022-0396/$ – see front matter
doi:10.1016/j.jde.2011.01.012

This paper is organized as follows. In this section, we introduce some notations and the functional framework. Some results concerning the weighted Sobolev spaces and the spaces of traces are recalled. In Sections 2 and 3, our work is focused on the div-grad-curl operators and elliptic problems in the whole space. After the case of the whole space, we then pass to the one of the half-space. Results in the half-space are presented in Section 4 (the div-grad operators), Section 5 (vector potentials) and in the last section of this paper (elliptic problems).

In this paper, we use bold type characters to denote vector distributions or spaces of vector distributions with n components and $C > 0$ usually denotes a generic positive constant that may depend on the dimension n, the exponent p and possibly other parameters, but never on the functions under consideration. For any real number $1 < p < \infty$, we take p' to be the Hölder conjugate of p. Let Ω be an open subset in the n-dimensional real Euclidean space. A typical point $\boldsymbol{x} \in \mathbb{R}^n$ is denoted by $\boldsymbol{x} = (\boldsymbol{x}', x_n)$, where $\boldsymbol{x}' = (x_1, x_2, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$. Its distance to the origin is denoted by $r = |\boldsymbol{x}| = (x_1^2 + \cdots + x_n^2)^{1/2}$. Let $\overline{\mathbb{R}^n_+}$ denote the closure of the upper half-space $\mathbb{R}^n_+ = \{\boldsymbol{x} \in \mathbb{R}^n;\ x_n > 0\}$. In the half-space, its boundary is defined by $\Gamma = \{\boldsymbol{x} \in \mathbb{R}^n;\ x_n = 0\} \equiv \mathbb{R}^{n-1}$. In order to control the behavior at infinity of our functions and distributions, we use for basic weight the quantity $\rho = \rho(r) = 1 + r$, which is equivalent to r at infinity. We define $\mathcal{D}(\Omega)$ to be the linear space of infinite differentiable functions with compact support on Ω. Now, let $\mathcal{D}'(\Omega)$ denote the dual space of $\mathcal{D}(\Omega)$, the space of distributions on Ω. For any $q \in \mathbb{N}$, $\mathscr{P}_q$ stands for the space of polynomials of degree $\leqslant q$. If q is a strictly negative integer, we set by convention $\mathscr{P}_q = \{0\}$. Given a Banach space B, with dual space B' and a closed subspace X of B, we denote by $B' \perp X$ (or more simply $X^\perp$, if there is no ambiguity as to the duality product) the subspace of B' orthogonal to X, i.e.

$$B' \perp X = X^\perp = \{ f \in B' \mid \forall v \in X,\ \langle f, v \rangle = 0 \} = (B/X)'.$$

The space $X^\perp$ is called the polar space of X in B' and is also denoted by X°. We also introduce the space

$$\boldsymbol{\mathcal{V}}(\Omega) = \{ \boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\Omega),\ \operatorname{div} \boldsymbol{\varphi} = 0 \}.$$

In this paper, we want to consider some particular weighted Sobolev spaces (see [4,5]). The open set Ω will denote the whole space or the half-space. We begin by defining the space

$$W_0^{1,p}(\Omega) = \left\{ u \in \mathcal{D}'(\Omega),\ \frac{u}{w_1} \in L^p(\Omega),\ \nabla u \in \mathbf{L}^p(\Omega) \right\},$$

where

$$w_1 = 1 + r \quad \text{if } p \neq n \quad \text{and} \quad w_1 = (1+r)\ln(2+r) \quad \text{if } p = n.$$

This space is a reflexive Banach space when endowed with the norm:

$$\|u\|_{W_0^{1,p}(\Omega)} = \left(\left\| \frac{u}{w_1} \right\|^p_{L^p(\Omega)} + \|\nabla u\|^p_{\mathbf{L}^p(\Omega)} \right)^{1/p}.$$

We also introduce the space

$$W_0^{2,p}(\Omega) = \left\{ u \in \mathcal{D}'(\Omega),\ \frac{u}{w_2} \in L^p(\Omega),\ \frac{\nabla u}{w_1} \in \mathbf{L}^p(\Omega),\ D^2 u \in \mathbf{L}^p(\Omega) \right\},$$

where

$$w_2 = (1+r)^2 \quad \text{if } p \notin \left\{ \frac{n}{2}, n \right\} \quad \text{and} \quad w_2 = (1+r)^2 \ln(2+r), \quad \text{otherwise},$$