J. Math. Anal. Appl. 438 (2016) 59–72

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Singularity formation for the compressible Euler equations with general pressure law

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ARTICLE INFO

Article history:
Received 17 September 2015
Available online 4 February 2016
Submitted by D. Wang

Keywords:
Conservation laws
Compressible Euler equations
General pressure law
Singularity formation
Large data

ABSTRACT

In this paper, the singularity formation of classical solutions for the compressible Euler equations with general pressure law is considered. The gradient blow-up of classical solutions is shown without any smallness assumption by delicate analysis of decoupled Riccati type equations. The proof also relies on a new estimate for the upper bound of the density.

1. Introduction

We consider the one dimensional compressible Euler equations in Lagrangian coordinates:

$$\begin{cases} \tau_t - u_x = 0, \\ u_t + p_x = 0, \\ \left(e + \frac{u^2}{2}\right)_t + (up)_x = 0, \end{cases} \tag{1.1}$$

where x is the space variable, t is the time variable, u is the velocity, ρ is the density, $\tau = \rho^{-1}$ is the specific volume, p is the pressure, and e is the internal energy. Due to the second law of thermodynamics, τ, p and e are not independent, the relation among them being determined by the state equation (*cf.* [9]). Normally, another physical quantity entropy S is considered, which formulates the state equation as $p = p(\tau, S)$. For C^1 solutions, the third equation of (1.1) is equivalent to the conservation of entropy (*cf.* [17]):

$$S_t = 0. \tag{1.2}$$

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http://dx.doi.org/10.1016/j.jmaa.2016.02.001

Apparently, (1.2) shows that S is just a function of x. The general pressure law we consider in this paper is

$$p = p(\tau, S) = p(\tau, S(x)). \tag{1.3}$$

Then the system (1.1) becomes

$$\begin{cases} \tau_t - u_x = 0, \\ u_t + p(\tau, S(x))_x = 0. \end{cases} \tag{1.4}$$

We consider the classical solution of initial value problem for (1.4) with initial data

$$\tau(x, t = 0) = \tau_0(x), \quad u(x, t = 0) = u_0(x).$$

The compressible Euler equations is one of the most important physical models for systems of hyperbolic conservation laws. It is well known that shock waves are typically formed in finite time and the analysis of the system is difficult because of the lack of regularity. The singularity formation for both the small initial data problem and the large initial data problem has long been a very serious issue for systems of conservation laws. The well-posedness theory for systems of hyperbolic conservation laws may be found in [1,9,10,18].

When initial data is small, the singularity formation has been well studied for decades. Lax [12] proved that the singularity forms in finite time for general systems of strictly hyperbolic conservation laws with two unknowns with some initial compression. For general systems of conservation laws, [11,13–15] provide fairly complete results for small data. Specifically, these results prove that the shock formation happens in finite time in any truly nonlinear characteristic field if the initial data includes compression.

However, the large data singularity formation theory has been finally established in very recent papers [4,7] for isentropic Euler equations with γ-law pressure ($p = K_1\tau^{-\gamma}$) and the full compressible Euler equations of polytropic ideal gas ($p = K_2 e^{S/c_\tau}\tau^{-\gamma}$), where K_1, K_2 are positive constants and $\gamma > 1$ is the adiabatic gas constant. The key point in proving the finite time shock formation for large solution is to have sharp global upper and lower bounds for the density. More precisely, if we restrict our consideration on singularity formation for the full compressible Euler equations, a uniform upper bound of the density is needed for any $\gamma > 1$, but a time-dependent lower bound of the density is needed only for the most physical case $1 < \gamma < 3$ (*cf.* [2]). A uniform upper bound of the density for γ-law pressure case has been found by [7] which directs to a resolution of shock formation when $\gamma \geq 3$. The singularity formation problem when $1 < \gamma < 3$ was finally resolved by [4], in which the authors proved a crucial time-dependent lower bound estimate for the density. Later on, the time-dependent lower bound of the density is improved to its optimal order $O(1/t)$ in [3].

Nevertheless, for the full compressible Euler equations with general pressure law, the singularity formation results for non-isentropic case are still not satisfied when the smallness assumption on the initial data is removed. In fact, a complete finite time gradient blow-up result has been showed in [4] when entropy S is a given constant. Furthermore, [6] provides a singularity formation result for non-isentropic general pressure law case. Unfortunately, in [6], there are still several a priori conditions on the pressure function which are not automatically satisfied for gas dynamics. The target of this paper is to establish a better singularity formation result of non-isentropic Euler equations without such kind of a priori assumptions. The key idea is to establish a uniform upper bound estimate for the density, which lacks in the general pressure law case previously. In this case, the lower bound of the density is redundant. Our proof relies on a careful study of decoupled Riccati type ordinary differential equations on gradient variables which was provided in [6]. Using our new estimates, we can get a constant lower bound on coefficients of the Riccati type equations, and the quadratic nonlinearity implies the derivatives must blow-up in finite time.

Through out this paper, we need to propose the following assumptions on the pressure: there exists a positive function $m = m(S)$, positive constants A, $k > 1$, k_1, k_2 and l_i $(i = 1, 2, \cdots, 8)$ such that, for $\tau \in (0, +\infty)$,