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Abstract Swiss cheese space and classicalisation of Swiss cheeses

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ABSTRACT

Swiss cheese sets are compact subsets of the complex plane obtained by deleting a sequence of open disks from a closed disk. Such sets have provided numerous counterexamples in the theory of uniform algebras. In this paper, we introduce a topological space whose elements are what we call "abstract Swiss cheeses". Working within this topological space, we show how to prove the existence of "classical" Swiss cheese sets (as discussed in [6]) with various desired properties. We first give a new proof of the Feinstein–Heath classicalisation theorem [6]. We then consider when it is possible to "classicalise" a Swiss cheese while leaving disks which lie outside a given region unchanged. We also consider sets obtained by deleting a sequence of open disks from a closed annulus, and we obtain an analogue of the Feinstein–Heath theorem for these sets. We then discuss regularity for certain uniform algebras. We conclude with an application of these techniques to obtain a classical Swiss cheese set which has the same properties as a non-classical example of O'Farrell [5].

1. Introduction

Throughout, we use the term *compact plane set* to mean a non-empty, compact subset of the complex plane. Let X be a compact plane set. Then $C(X)$ denotes the set of all continuous, complex-valued functions on X, and $R(X)$ denotes the set of those functions $f \in C(X)$ which can be uniformly approximated on X by rational functions with no poles on X. Both $R(X)$ and $C(X)$ are uniform algebras on X. We refer the reader to [1,2,8,13] for further definitions and background concerning uniform algebras and Banach algebras.

A Swiss cheese set is a compact subset of $\mathbb{C}$ obtained by deleting a sequence of open disks from a closed disk. Such sets have been used as examples in the theory of uniform algebras and rational approximation.

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Swiss cheese sets were introduced by Roth [12], where she gave the first known example of a compact plane set X such that $R(X) \neq C(X)$ but X has empty interior. Since then there have been numerous applications of Swiss cheese sets in the literature.

One notable example of a Swiss cheese construction is due to McKissick [11]. He gave an example of a Swiss cheese set X such that $R(X)$ is regular but $R(X) \neq C(X)$. (We will define regularity in Section 7.) The sequence of open disks used to construct this Swiss cheese set may touch or overlap, which means that the set X might have undesirable topological properties. To improve the topological properties of the resulting Swiss cheese set, while preserving the properties of the uniform algebra, a process that we call *classicalisation* was developed [6].

We may consider a pair consisting of a closed disk and a collection of open disks in the plane, from which we obtain the desired Swiss cheese set (see Definition 2.1 below). We call such a pair a *Swiss cheese* and say it is *classical* if the collection of open disks and the complement of the closed disk have pairwise disjoint closures and the sum of the radii of all open disks is finite. Note that, in the literature, the term 'Swiss cheese' traditionally refers to what we call a Swiss cheese set. Feinstein and Heath [6] considered Swiss cheeses in which the sum of the radii of the open disks is strictly less than the radius of the larger, closed disk. They proved, using Zorn's lemma, that for such a Swiss cheese, the associated Swiss cheese set contains a Swiss cheese set associated to a classical Swiss cheese. Later, Mason [10] gave a proof of this theorem using transfinite induction.

Classical Swiss cheese sets have many desirable topological properties. For example, Dales and Feinstein [3] proved that given two points x, y in a classical Swiss cheese set there is a rectifiable path connecting x, y and such that the length of this path is no more than $\pi|x-y|$; in fact, the constant π can be replaced by $\pi/2$ here. After this observation it is easy to see that a classical Swiss cheese set is path connected (and hence connected), locally path connected (and hence locally connected), and uniformly regular, as defined in [3]. Also as a consequence of connectedness, we see that a classical Swiss cheese set cannot have any isolated points. In [6] it was noted that every classical Swiss cheese set with empty interior is homeomorphic to the Sierpiński carpet as a consequence of a theorem of Whyburn [14].

Browder [1] notes that if X is a classical Swiss cheese set then $R(X)$ is essential (see also [6]). In particular, $R(X) \neq C(X)$, as originally proved by Roth [12]. It follows from the Hartogs–Rosenthal theorem that X must have positive area. A direct proof that every classical Swiss cheese set has positive area is due to Allard, as outlined in [1, pp. 163–164].

Where existing examples of Swiss cheese sets in the literature are not classical, it is of interest to construct classical Swiss cheese sets which solve the same problems. As part of a general classicalisation scheme, we discuss some new techniques for constructing such classical Swiss cheese sets.

In this paper we consider what we call *abstract Swiss cheeses*, which are sequences of pairs consisting of a complex number and a non-negative real number. Each pair in this sequence corresponds to a centre and radius of a disk in the plane. We give the set of all abstract Swiss cheeses a natural topology and use this topology to give a new proof of the Feinstein–Heath theorem. We show that, under some conditions, we can classicalise Swiss cheese sets while only changing open disks which lie in certain regions. We prove an analogue of the Feinstein–Heath theorem for annuli. We give some results regarding regularity of $R(X)$ for unions of compact plane sets, which will be used in the final section. Finally, we give an example of the application of a combination of these results to construct an example of a classical Swiss cheese set X such that $R(X)$ is regular and admits a non-degenerate bounded point derivation of infinite order (as defined in Section 8), which improves an example of O'Farrell [5]. This fits into our general classicalisation scheme.

2. Swiss cheeses and abstract Swiss cheese space

We denote the set of all non-negative real numbers by $\mathbb{R}^+$, the set of positive integers by $\mathbb{N}$ and the set of all non-negative integers by $\mathbb{N}_0$. Let $a \in \mathbb{C}$ and let $r > 0$. We denote the open disk of radius r and centre