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Harnack inequalities for supersolutions of fully nonlinear elliptic difference and differential equations

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A B S T R A C T

We present a new Harnack inequality for non-negative discrete supersolutions of fully nonlinear uniformly elliptic difference equations on rectangular lattices. This estimate applies to all supersolutions and has the Harnack constant depending on the graph distance on lattices. For the proof we modify the proof of the weak Harnack inequality. Applying the same idea to elliptic equations in a Euclidean space, we also derive a Harnack type inequality for non-negative viscosity supersolutions.

1. Introduction

We consider fully nonlinear, non-homogeneous second order equations of the form

$$F(D^2u) = f(x) \tag{1.1}$$

with a uniformly elliptic operator F. A typical statement of the Harnack inequality is that there exists a constant $C > 0$ such that the inequality

$$\max_U u \leqq C \left\{ \min_U u + \|f\|_{L^n(V)} \right\} \tag{1.2}$$

holds for every non-negative solution u of (1.1) in V. Here V is a set which is (enough) larger than U, and n represents the dimension of space. One of well-known proofs of the Harnack inequality is a combination of a *weak Harnack inequality*, which asserts that, for some $p > 0$,

$$\|u\|_{L^p(U)} \leqq C \left\{ \min_U u + \|f\|_{L^n(V)} \right\}$$

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holds for every non-negative supersolution u, and a *local maximum principle* (or a *mean value inequality*):

$$\max_U u \leqq C \left\{ \|u\|_{L^q(U)} + \|f\|_{L^n(V)} \right\}$$

for subsolutions u, where $q > 0$ is arbitrary. These estimates are well-known in the continuum case where (1.1) is studied as a partial differential equation in $\mathbf{R}^n$; for instance, the reader is referred to [9, Chapter 9] for linear equations and to [3, Chapter 4] for fully nonlinear equations. The corresponding results are also obtained in the discrete case when we study (1.1) as a difference equation on lattices. In [13] the Harnack inequality for elliptic difference equations is derived via the weak Harnack inequality and the local maximum principle. See also [15] for the parabolic case and [16,17] for general meshes.

The main goal of this paper is to show that, in the discrete case, a modified proof of the weak Harnack inequality implies a new type of the Harnack inequality for discrete supersolutions to (1.1) on rectangular lattices (Theorem 4.1). Our proof is direct and simple in the sense that we do not need the local maximum principle. Accordingly, the resulting estimate is valid for every supersolution which is not necessarily a subsolution. This is a difference from the literature.

It seems that importance of analysis on various non Euclidean spaces has been increasing. In fact, one of motivations of this work comes from partial differential equations posed on a discrete space such as graphs, whose application include, e.g., stochastic controls, mean field games, social networks, and so on. (See [18] and references therein.) In the recent paper [18], nonlinear elliptic partial differential equations are studied on a finite graph, and uniqueness and existence results of solutions are obtained. Our setting on rectangular lattices is a special case of [18]. A flavor of analysis for graph spaces could be seen in the present paper; we use the graph distance $\rho(x)$ and the ball B_r with respect to this distance (Section 3). They appear in the statement of our main theorem (Theorem 4.1). It is our surprise that Harnack estimate holds for supersolutions if we consider a discrete space. This enables us to control more functions since our result does not rely on a subsolution property of solutions.

It turns out that our Harnack constant, C in (1.2), depends on the graph distance on lattices. Due to this, passing to limits in our Harnack inequality does not imply the continuum Harnack inequality since the Harnack constant C goes to infinity when the mesh size tends to 0 (Remarks 3.4 and 4.2). Here it is worth mentioning that such reconstruction of the continuum Harnack inequality should not be possible since (1.2) does not hold even for the Laplace equation if we do not require u to be a subsolution; see Example 5.3 for the counter-example. Contribution of this paper is a discovery of a Harnack estimate for functions (supersolutions) belonging to a wider class which are excluded in a study of a convergent scheme. This is our first attempt to give a priori estimate for discrete solutions, and the extension of the result to more general lattices rather than rectangular lattices is one of interesting our future problems.

In the proof of the weak Harnack inequality for fully nonlinear equations of the continuum case ([2,3]), we take a radially symmetric and increasing supersolution ϕ of the Pucci equation

$$\mathcal{P}^-(D^2\phi) = -\xi(x). \tag{1.3}$$

Here $\mathcal{P}^-$ is a Pucci operator (see (5.2) or (2.2) for definition) and ξ is a non-negative, continuous function whose support is contained in a small ball centered at the origin. Such a function ϕ is often called a *barrier function*. In the discrete case, we are able to construct the barrier function so that ξ is non-zero only at the origin (Lemma 3.1 and Remark 3.2). In other words, its support is only one point. This is a crucial difference from the continuous case, and this enables a pointwise estimate for supersolutions of difference equations. In our proof of the Harnack inequality, we translate the barrier function so that its minimum point, which originally lies at the origin, comes to a maximum point of the supersolution u of (1.1). As a result, we obtain the Harnack inequality without discussing the local maximum principle.