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A Korteweg–de Vries type of fifth-order equations on a finite domain with point dissipation

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ABSTRACT

The paper discusses initial value problem of a Korteweg–de Vries type of fifth-order equation

$$w_t + w_{xxx} - w_{xxxxx} - \sum_{j=1}^{n} a_j w^j w_x = 0\,, \qquad w(x,0) = w_0(x)$$

posed on a periodic domain $x \in [0, 2\pi]$ with periodic boundary conditions $w_{ix}(0,t) = w_{ix}(2\pi,t)$, $i = 0, 2, 3, 4$ and an L^2-stabilizing feedback control law $w_x(2\pi,t) = \alpha w_x(0,t) + (1-\alpha)w_{xxx}(0,t)$ where n is a fixed positive integer, a_j, $j = 1, 2, \cdots, n$, α are real constants, and $|\alpha| < 1$. It is shown that for $w_0(x) \in H^1_\alpha(0, 2\pi)$ with the boundary conditions described above, the problem is locally well-posed for $w \in C([0,T]; H^1_\alpha(0,2\pi))$ with a conserved volume of w, $[w] = \int_0^{2\pi} w(x,t)dx$. Moreover, the solution with small initial condition exists globally and approaches to $[w_0(x)]/(2\pi)$ as $t \to +\infty$.

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1. Introduction

In this paper, we study the solutions of the following general nonlinear problem

$$w_t + w_{xxx} - w_{xxxxx} - \sum_{j=1}^{n} a_j w^j w_x = 0\,, \qquad w(x,0) = w_0(x) \tag{1.1}$$

posed on a domain $x \in [0, 2\pi]$. Here, n is a positive integer and a_j, $j = 1, 2, \cdots, n$ are real constants. This type of equations has been used to model traveling gravity–capillary waves on a two-dimensional fluid flow

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of constant depth. When the traveling speed of the wave is near its critical value and the surface tension is not near its critical value, the famous Korteweg–de Vries (KdV) equation

$$w_t + w_{xxx} - aww_x = 0 \tag{1.2}$$

can be derived for modeling the propagation of surface waves [15,31]. However, if the surface tension is also near its critical value, the KdV equation is no longer valid and the equation (1.1) with $a_1 \neq 0$ and $a_j = 0$, $j = 2, 3, \cdots, n$, called a fifth-order KdV equation or Kawahara equation, can be derived [11]. In some special cases, the coefficient a_1 may be zero for two-layer fluid systems and then the equation (1.1) with $a_2 \neq 0$ and $a_j = 0$, $j = 1, 3, 4, \cdots, n$, called a modified fifth-order KdV equation or modified Kawahara equation, can be obtained (see [4] for corresponding modified KdV equation). Here, we are interested in stabilization problem for a general equation (1.1) under some boundary conditions.

The KdV equation has been studied extensively during last half century (see [2,3,8,12,28] and references therein) and its control problems have been also discussed since 1990s. One type of control problem for (1.2) was first discussed by Russell and Zhang [25,26] using the boundary conditions

$$w(0,t) = w(2\pi,t)\,, \quad w_{xx}(0,t) = w_{xx}(2\pi,t)\,, \quad w_x(2\pi,t) = \alpha w_x(0,t) \tag{1.3}$$

with $|\alpha| < 1$ ($\alpha = -1/2$ is a singular case and was studied in [30]). In control theory, it is called a closed-loop control process that generally refers to control synthesis via some kind of state feedback and is mainly concerned with achieving asymptotic stability of an equilibrium or invariant set. It is straightforward to see that a constant state is an equilibrium state for both (1.1) and (1.2). In [26], it is shown that for small initial data, the solution of (1.2) with boundary conditions (1.3) always exists and goes to a constant state $\frac{1}{2\pi}\int_0^{2\pi} w_0(x)dx$ as $t \to +\infty$. Moreover, a singular case was studied in [30] where the method used in [26] fails to produce the asymptotic stability result for (1.2) and (1.3). The first stability result of the KdV equation with periodic boundary conditions for large initial data was given in [17], while similar results with some dissipative boundary conditions were discussed in [18,20,21]. Other types of control problems for (1.2) can be found in [14,27,22]. In particular, a review paper [22] for this type of problems is recommended.

The well-posedness of pure initial value problem for (1.1) in $\mathbb{R}$ with $a_j = 0$, $j = 2, \cdots, n$, i.e., the fifth-order KdV equation, has been discussed in [9,10]. The controllability problems of (1.1) for this case using nonhomogeneous terms in the equation or boundary conditions have been studied in [32,33]. They showed the boundary smoothing properties for the linear fifth-order KdV equation with boundary conditions [33] and local controllability and stabilization of the nonlinear fifth-order KdV equation on a periodic domain with an internal control [32]. Here, we only study the local and global well-posedness of (1.1) and the asymptotic stability of small solutions as $t \to +\infty$ using *a closed-loop point dissipation process* (general discussions on such control problems can be found in [6,23]). To design a dissipation mechanism for (1.1), we multiply both sides of (1.1) by $w(x,t)$ and integrate it from zero to 2π. If we impose periodic boundary conditions

$$w_{kx}(0,t) = w_{kx}(2\pi,t) \qquad \text{for} \quad k = 0, 2, 3, 4 \tag{1.4}$$

where $w_{kx} = \frac{\partial^k w}{\partial x^k}$, then we can obtain an identity for the solution of (1.1),

$$\frac{d}{dt}\left(\int_0^{2\pi} w^2(x,t)dx\right) = \big(w_x(2\pi,t) - w_x(0,t)\big)\cdot\big(w_x(2\pi,t) + w_x(0,t) - 2w_{3x}(0,t)\big). \tag{1.5}$$

When we let