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The borderline of the short-range condition for the repulsive Hamiltonian

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ABSTRACT

We consider quantum systems described by the Schrödinger equation equipped with a so-called repulsive part. In this paper, we find a counterexample to the slow decaying interaction potential such that the wave operators do not exist, and each a conclusion regarding the borderline between the short-range and long-range.

1. Introduction

In this paper, we consider a scattering problem for the Schrödinger equation equipped with a repulsive part. The pair of Hamiltonians under consideration is given by

$$H_0 = p^2 - x^2, \quad H = H_0 + V \tag{1.1}$$

acting on $L^2(\mathbb{R}^n)$, where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is the position of the particle, $p = (-i\partial_{x_1}, \ldots, -i\partial_{x_n}) = -i\nabla$ is the momentum operator and x^2 means $|x|^2$. In the free Hamiltonian H_0, $-x^2$ is called the repulsive part. If this part is replaced by $+x^2$, then this operator is the harmonic oscillator and the particle does not scatter in this case. However, in the repulsive case, the particle can scatter with surprising velocity. In this sense, the sign in front of x^2 is very effective in scattering theory. It is known that H_0 is essentially self-adjoint on the domain associated with the harmonic oscillator. The spectrum of H_0 is purely absolutely continuous and is located all over $\mathbb{R}$ (see Bony et al. [1]). On the other hand, in the full Hamiltonian H, the interaction potential $V = V(x)$ is a real-valued multiplication operator. Of course, we assume that $V(x)$ vanishes when $|x|$ is sufficiently large because V represents the interaction. In this paper, we only treat the

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case $V \in L^\infty(\mathbb{R}^n)$, and therefore H is also essentially self-adjoint. We represent the self-adjoint extensions by H_0 and H again.

In the case where the free Hamiltonian is $p^2 = -\Delta$, if we write the decay condition on V as

$$|V(x)| \lesssim \langle x \rangle^{-\rho} \tag{1.2}$$

with $\rho > 0$ and $\langle x \rangle = (1+x^2)^{1/2}$, then it is well known that, if $\rho > 1$, then the wave operators exist, and if $\rho \leqslant 1$, then wave operators do not exist (Reed and Simon [7, Corollary 2 of Theorem XI.71]). That is to say, the borderline between the short-range and long-range is $\rho = 1$. The classical trajectory of the particle in the dynamics of the free Schrödinger equation has order $x(t) = O(t)$ as $t \to \infty$. The Cook–Kuroda method claims that if

$$\int_1^\infty dt\ \|Ve^{-itp^2}\phi\| < \infty \tag{1.3}$$

for $\phi \in L^2(\mathbb{R}^n)$, then the wave operators s-$\lim_{t\to\pm\infty} e^{it(p^2+V)}e^{-itp^2}$ exist. This is because

$$\begin{aligned} \|(e^{it_1(p^2+V)}e^{-it_1p^2} - e^{it_2(p^2+V)}e^{-it_2p^2})\phi\| &\leqslant \int_{t_2}^{t_1} dt\ \|\partial_t(e^{it(p^2+V)}e^{-itp^2})\phi\| \\ &= \int_{t_2}^{t_1} dt\ \|Ve^{-itH_0}\phi\| \longrightarrow 0 \end{aligned} \tag{1.4}$$

as $t_1, t_2 \to \infty$. Therefore, we can formally verify the borderline by substituting the classical order $x(t) = O(t)$ for $V(x)$,

$$\int_1^\infty dt\ \|Ve^{-itp^2}\phi\| = \int_1^\infty dt\ \|V(x(t))e^{-itp^2}\phi\| \lesssim \int_1^\infty dt\ t^{-\rho}, \tag{1.5}$$

because of the decay assumption (1.2). This estimate is very rough and formal, but the right-hand side of (1.5) is bounded if and only if $\rho > 1$.

In the case of the Stark effect, that is, $H_0^S = p^2 - E \cdot x$ with $E \in \mathbb{R}^n \setminus \{0\}$, the classical trajectory has order $x(t) = O(t^2)$ as $t \to \infty$ by solving the Newton equation $\ddot{x}(t)/2 = E$. By the rough estimate

$$\int_1^\infty dt\ \|Ve^{-itH_0^S}\phi\| \lesssim \int_1^\infty dt\ t^{-2\rho}, \tag{1.6}$$

we expect that the borderline will be $\rho = 1/2$. In fact, an affirmative answer for this was given by Ozawa [6]. By constructing a concrete counterexample for the potential V such that, for $H^S = H_0^S + V$, the wave operators s-$\lim_{t\to\pm\infty} e^{itH^S}e^{-itH_0^S}$ do not exist, Ozawa [6] determined that the borderline has to be $\rho = 1/2$.

In the case of the repulsive pair of Hamiltonians, the classical trajectory is given by solving the Newton equation $\ddot{x}(t)/2 = 2x(t)$, and thus we have $x(t) = O(e^{2t})$ as $t \to \infty$. From the analogies before, when we impose the decay condition on V by

$$|V(x)| \lesssim (\log\langle x \rangle)^{-\rho} \tag{1.7}$$