J. Math. Anal. Appl. 438 (2016) 351–372

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Truncated spectral regularization for an ill-posed nonhomogeneous parabolic problem

Ajoy Jana, M. Thamban Nair *

Department of Mathematics, IIT Madras, India

ARTICLE INFO

Article history:
Received 25 July 2015
Available online 9 February 2016
Submitted by B. Kaltenbacher

Keywords:
Ill-posed problems
Parabolic equations
Regularization
Parameter choice
Semigroup

ABSTRACT

The non homogeneous backward Cauchy problem $u_t + Au = f(t)$, $u(\tau) = \phi$ for $0 \le t < \tau$ is considered, where A is a densely defined positive self-adjoint unbounded operator on a Hilbert space H with $f \in L^1([0,\tau], H)$ and $\phi \in H$ is known to be an ill-posed problem. A truncated spectral representation of the mild solution of the above problem is shown to be a regularized approximation, and error analysis is considered when both ϕ and f are noisy. Error estimates are derived under appropriate choice of the regularization parameter. The results obtained unify and generalize many of the results available in the literature.

1. Introduction

Let H be a Hilbert space and $A : D(A) \subset H \to H$ be a densely defined positive self-adjoint unbounded operator. For $\tau > 0$, $\phi \in H$ and $f \in L^1([0,\tau], H)$, consider the problem of solving the *final value problem*, denoted briefly as FVP,

$$u_t + Au = f(t), \qquad 0 \le t < \tau \tag{1.1}$$

$$u(\tau) = \phi. \tag{1.2}$$

Here, $L^1([0,\tau], H)$ denotes the space of all H-valued integrable functions on $[0,\tau]$, i.e., $g \in L^1([0,\tau], H)$ if and only if $g : [0,\tau] \to H$ is measurable and

$$\int_0^\tau \|g(t)\| dt < \infty.$$

* Corresponding author.
E-mail addresses: janaajoy340@gmail.com (A. Jana), mtnair@iitm.ac.in (M.T. Nair).

http://dx.doi.org/10.1016/j.jmaa.2016.01.069
0022-247X/

The problem is to find a function $u : [0, \tau] \to H$ which is differentiable and satisfies the equations (1.1) and (1.2). It is well known that the above FVP is ill-posed (cf. Goldstein [6]). Therefore, in order to obtain stable approximate solutions for (1.1)–(1.2), some regularization method has to be employed. A particular case of the above FVP which has got wide applications in science and engineering is the backward heat conduction problem (BHCP) in which the Hilbert space H is the space $L^2(\Omega)$, where Ω is a domain in $\mathbb{R}^k$ for some $k \in \mathbb{N}$, and $-A = \Delta$, the Laplacian operator in $L^2(\Omega)$ (see Isakov [7], Nair [10]).

The homogeneous FVP, that is, when $f = 0$, has been studied by many authors using different approaches. Many of them have used the *quasi-reversibility method*, introduced by Lattes and Lions [8]. The main idea of this method is to consider a perturbed form of the operator A (see e.g., Miller [9], Showalter [12] and Boussetila and Rebbani [2]). Another approach to study the homogeneous FVP considered by some authors is by perturbing the final value; such method is called *quasi-boundary value method* (see, e.g. Clark and Oppenheimer [3], Denche and Bessila [4], Denche and Djezzar [5]). Clark and Oppenheimer, Denche and Bessila have restricted their study of quasi-boundary value method when operator A is having discrete spectrum. In [1], Boussetila and Rebbani have studied homogeneous FVB by perturbing the final value as well as the operator A.

We may recall from semigroup theory (cf. [11]) that if $u(\cdot)$ is a solution of the equation

$$u_t + Au = f(t), \qquad 0 < t \le \tau,$$

then it has the representation

$$u(t) = S(t)\phi_0 + \int_0^t S(t-s)f(s)ds$$

where $\phi_0 = u(0)$ and $\{S(t) : t \ge 0\}$ is the C_0 semigroup generated by $-A$. In fact,

$$S(t) = e^{-tA} := \int_0^\infty e^{-t\lambda} dE_\lambda,$$

where $\{E_\lambda : \lambda \ge 0\}$ is the resolution of identity of A, and $\{e^{-tA} : t \ge 0\}$ is a differentiable semigroup (cf. [11]). With the above notation,

$$u(t) = \int_0^\infty e^{-t\lambda} dE_\lambda \phi_0 + \int_0^t \left(\int_0^\infty e^{-(t-s)\lambda} dE_\lambda f(s) \right) ds.$$

Note that the above representation is meaningful whenever $f \in L^1([0, \tau], H)$, and in that case $u : [0, \tau] \to H$ defined by

$$u(t) = \int_0^\infty e^{-t\lambda} dE_\lambda \phi_0 + \int_0^t \left(\int_0^\infty e^{-(t-s)\lambda} dE_\lambda f(s) \right) ds \tag{1.3}$$

is called the *mild solution* of the initial value problem (IVP)

$$u_t + Au = f(t), \qquad 0 < t \le \tau, \tag{1.4}$$

$$u(0) = \phi_0 \tag{1.5}$$