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Existence of global strong solution for Korteweg system with large infinite energy initial data

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ABSTRACT

This work is devoted to the study of the initial boundary value problem for a general isothermal model of capillary fluids derived by J.E. Dunn and J. Serrin (1985) (see [18]), which can be used as a phase transition model. We aim at proving the existence of local and global (under a condition of smallness on the initial data) strong solutions with initial density $\ln\rho_0$ belonging to the Besov space $B^{\frac{N}{2}}_{2,\infty}$. It implies in particular that some classes of discontinuous initial density generate strong solutions. The proof relies on the fact that the density can be written as the sum of the solution ρ_L of the associated linear system and a remainder term $\bar{\rho}$; this last term is more regular than ρ_L provided that we have regularizing effects induced on the bilinear convection term. The main difficulty consists in obtaining new estimates of maximum principle type for the associated linear system; this is based on a characterization of the Besov space in terms of the semi-group associated with this linear system. We show in particular the existence of global strong solution for small initial data in $(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,\infty}\cap L^\infty)\times B^{\frac{N}{2}-1}_{2,\infty}$; it allows us to exhibit a family of large energy initial data when $N=2$ providing global strong solution. In conclusion we introduce the notion of quasi-solutions for the Korteweg's system (a tool which has been developed in the framework of the compressible Navier–Stokes equations [31,30,32,26,27]) which enables to obtain the existence of global strong solution with a smallness condition which is subcritical. Indeed we can deal with large initial velocity in $B^{\frac{N}{2}-1}_{2,1}$. As a corollary, we get global strong solution for highly compressible Korteweg system when $N\geq 2$. It means that for any large initial data (under an irrotational condition on the initial velocity) we have the existence of global strong solution provided that the Mach number is sufficiently large.

1. Introduction

We are concerned with compressible fluids endowed with internal capillarity. The model we consider originates from the XIXth century work by J.F. Van der Waals and D.J. Korteweg [44,37] and was actually

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derived in its modern form in the 1980s using the second gradient theory (see [18,35,43]). The first investigations begin with the Young–Laplace theory which claims that the phases are separated by a hypersurface and that the jump in the pressure across the hypersurface is proportional to the curvature of the hypersurface. The main difficulty consists in describing the location and the movement of the interfaces. Another major problem is to understand whether the interface behaves as a discontinuity in the state space (sharp interface) or whether the phase boundary corresponds to a more regular transition (diffuse interface, DI). The diffuse interface models have the advantage to consider only one set of equations in a single spatial domain (the density takes into account the different phases) which considerably simplifies the mathematical and numerical study (indeed in the case of sharp interfaces, we have to treat a problem with free boundary). Another approach corresponds to determining equilibrium solutions which classically consists of the minimization of the free energy functional. Unfortunately this minimization problem has an infinity of solutions, and many of them are physically irrelevant. In order to overcome this difficulty, J.F. Van der Waals in the XIX-th century was the first to add a term of capillarity in order to select the physically correct solutions. This theory is widely accepted as a thermodynamically consistent model for equilibria.

Korteweg-type models are based on an extended version of nonequilibrium thermodynamics, which assumes that the energy of the fluid not only depends on standard variables but also on the gradient of the density. Alternatively, another way to penalize the high density variations consists in applying a zero order but non-local operator to the density gradient (see [42,41,40]). For more results on non-local Korteweg system, we refer also to [10–13,23,24].

Let us now consider a fluid of density $\rho \geq 0$, velocity field $u \in \mathbb{R}^N$, we are now going to consider the so-called local Korteweg system which is a compressible capillary fluid model, it can be derived from a Cahn–Hilliard like free energy (see the pioneering work by J.E. Dunn and J. Serrin in [18] and also [1,8,21]). The conservation of mass and of momentum write:

$$\begin{cases} \dfrac{\partial}{\partial t}\rho + \mathrm{div}(\rho u) = 0, \\ \dfrac{\partial}{\partial t}(\rho u) + \mathrm{div}(\rho u \otimes u) - \mathrm{div}(2\mu(\rho)D(u)) - \nabla\big(\lambda(\rho))\mathrm{div}u\big) + \nabla P(\rho) = \mathrm{div}K, \end{cases} \tag{1.1}$$

where the Korteweg tensor reads as:

$$\mathrm{div}K = \nabla\big(\rho\kappa(\rho)\Delta\rho + \frac{1}{2}(\kappa(\rho) + \rho\kappa'(\rho))|\nabla\rho|^2\big) - \mathrm{div}\big(\kappa(\rho)\nabla\rho\otimes\nabla\rho\big). \tag{1.2}$$

κ is the coefficient of capillarity and is a regular function. The term $\mathrm{div}K$ allows to describe the variation of density at the interfaces between two phases, generally a mixture liquid-vapor. P is a general increasing pressure. $D(u) = \frac{1}{2}(\nabla u +^t \nabla u)$ defines the stress tensor, μ and λ are the two Lamé viscosity coefficients depending on the density ρ and satisfying:

$$\mu > 0 \quad \text{and} \quad 2\mu + N\lambda \geq 0.$$

We briefly recall the classical energy estimates for the system (1.1); let $\bar{\rho} > 0$ be a constant reference density (in what follows, we shall assume that $\bar{\rho} = 1$) and let Π be defined by:

$$\Pi(s) = s\Big(\int_{\bar{\rho}}^{s} \frac{P(z)}{z^2}dz - \frac{P(\bar{\rho})}{\bar{\rho}}\Big),$$

so that $P(s) = s\Pi'(s) - \Pi(s)$, $\Pi'(\bar{\rho}) = 0$. Multiplying the equation of momentum conservation in the system (1.1) by u and integrating by parts over $(0,t)\times\mathbb{R}^N$, we get the following estimate: