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On primal regularity estimates for set-valued mappings

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ABSTRACT

We prove several generalizations of the results in [6] for set-valued mappings. In some cases, we improve also the statements for single-valued mappings. Linear openness of the set-valued mapping in question is deduced from the properties of its suitable approximation. This approach goes back to the classical Lyusternik–Graves theorem saying that a continuously differentiable single-valued mapping between Banach spaces is linearly open around an interior point of its domain provided that its derivative at this point is surjective. In this paper, we consider approximations given by a graphical derivative, a contingent variation, a strict pseudo H-derivative, and a bunch of linear mappings.

1. Introduction

Metric regularity, linear openness and pseudo-Lipschitz property of the inverse are tree *equivalent* properties playing fundamental role in modern variational analysis and have been broadly covered in the recent monographs [3,9,18,22]. A survey on this topic together with a rich bibliography can be found in [16].

Banach's open mapping principle says that a continuous linear mapping between two Banach spaces is (linearly) open (at any point) if and only if it is surjective. In 1950, L.M. Graves generalized this statement proving that a single-valued mapping f acting between Banach spaces is (linearly) open at an interior point $\bar{x}$ of its domain provided that there exists a surjective continuous linear mapping A such that the difference $f - A$ is locally Lipschitz continuous at $\bar{x}$ with a sufficiently small Lipschitz modulus. This result remains true [5] when $\bar{x}$ is a boundary point of a closed convex subset K of a Banach space and the restriction of the approximating mapping A to K is open at $\bar{x}$. In variational analysis we work with mappings which may be non-smooth and also set-valued. Note that the mapping f in Graves' theorem is not necessarily

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differentiable at $\bar{x}$ but it is well approximated by one single-valued mapping around the point in question. K. Nachi and J.-P. Penot [19] extended this idea to a set-valued mapping F between Banach spaces by introducing a suitable derivative of F at the reference point which is supposed to be a (single-valued) continuous linear mapping. A different approach was used by B.H. Pourciau in [23] who proved that a Lipschitz-continuous single-valued mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$, with $m \leq n$, is linearly open at an interior point $\bar{x}$ of its domain provided that all the matrices in the Clarke's generalized Jacobian have full (row) rank. His result was extended to reflexive Banach spaces by D. Preiss and the second named author in [12] where f is approximated by a bounded convex set of continuous linear operators, and to general Banach spaces by Z. Páles in [21] under rather strong compactness assumption on the bunch of linear operators (with respect to the topology induced by the operator norm). Similarly as in the case of Graves' theorem, these results remain true [5,6] when $\bar{x}$ is a boundary point of the domain of f provided that it is closed and convex. Of course, there are many other ways how to approximate a set-valued mapping (for more details, see bibliographical comments following the statements in Section 3). In this paper, we focus on approximations given by a graphical derivative, or by a contingent variation, or by a strict pseudo H-derivative, or by a bunch of continuous linear mappings. We obtain generalizations of the results mentioned above.

The paper is organized as follows. In the next section, we provide a background from regularity theory. The statements therein will be used in Section 3 which contains all our results together with relevant bibliographical comments.

Notations and terminology When we write $f : X \to Y$ we mean that f is a (single-valued) mapping acting from X into Y while $F : X \rightrightarrows Y$ is a mapping from X into Y which may be set-valued. The set $\operatorname{dom} F := \{x : F(x) \neq \emptyset\}$ is the *domain* of F, the *graph* of F is the set $\operatorname{gph} F := \{(x, y) \in X \times Y : \ y \in F(x)\}$ and the *inverse* of F is the mapping $Y \ni y \longmapsto \{x \in X : \ y \in F(x)\} =: F^{-1}(y) \subset X$; thus $F^{-1} : Y \rightrightarrows X$. In any metric space $B(x, r)$ denotes the closed ball centered at x with a radius $r > 0$ and $\overset{\circ}{B}(x, r)$ is the corresponding open ball. B_X and S_X are respectively the *closed unit ball* and the *unit sphere* in a Banach space X. The *distance from a point* x to a subset C of a metric space (X, d) is $d(x, C) := \inf\{d(x, y) : \ y \in C\}$. We use the convention that $\inf \emptyset := +\infty$ and as we work with non-negative quantities we set $\sup \emptyset := 0$. If a set is singleton we identify it with its only element, that is, we write a instead of $\{a\}$. The symbol $\mathcal{L}(X, Y)$ denotes the space of all linear bounded operators from a Banach space X into a Banach space Y.

2. Background from regularity theory

Given two metric spaces X and Y, a set-valued mapping $F : X \rightrightarrows Y$ is called *open with a linear rate* near $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ if there are positive numbers c and ε such that

$$B(y, ct) \subset F(B(x, t)), \quad \text{whenever} \quad (x, y) \in \big(B(\bar{x}, \varepsilon) \times B(\bar{y}, \varepsilon)\big) \cap \operatorname{gph} F \quad \text{and} \quad t \in (0, \varepsilon). \tag{1}$$

The supremum of $c > 0$ such that (1) holds for some $\varepsilon > 0$ is called *rate of openness* (*rate* or *modulus of surjection*) of F near $(\bar{x}, \bar{y})$ and is denoted by $\operatorname{sur} F(\bar{x}, \bar{y})$. The mapping $F : X \rightrightarrows Y$ with $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ is said to be *metrically regular* near $(\bar{x}, \bar{y})$ if there are $\kappa > 0$ and neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$d\big(x, F^{-1}(y)\big) \leq \kappa d(y, F(x)) \quad \text{for all} \quad (x, y) \in U \times V. \tag{2}$$

The infimum of all $\kappa > 0$ such that (2) holds for some neighborhoods U and V is called the *rate* or *modulus of metric regularity* of F near $(\bar{x}, \bar{y})$ and is denoted by $\operatorname{reg} F(\bar{x}, \bar{y})$. It is well known that $\operatorname{reg} F(\bar{x}, \bar{y}) \cdot \operatorname{sur} F(\bar{x}, \bar{y}) = 1$ always holds under convention that $0 \cdot \infty = 1$ (see [16] for history of this equality). If $f : X \to Y$ is a single-valued mapping, then we write $\operatorname{sur} f(\bar{x})$ and $\operatorname{reg} f(\bar{x})$ instead of $\operatorname{sur} f\big(\bar{x}, f(\bar{x})\big)$ and $\operatorname{reg} f\big(\bar{x}, f(\bar{x})\big)$, respectively. An $A \in \mathcal{L}(X, Y)$ is metrically regular at any point if and only if it is surjective; therefore we write $\operatorname{sur} A$ and $\operatorname{reg} A$ only.