J. Math. Anal. Appl. 436 (2016) 131–137

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Phase retrieval versus phaseless reconstruction ☆

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A R T I C L E   I N F O

Article history:
Received 27 June 2015
Available online 2 December 2015
Submitted by E. Saksman

Keywords:
Phase retrieval
Phaseless reconstruction
Complement property
Projections
Weak phase retrieval
Frame theory

A B S T R A C T

In 2006, Balan, Casazza, and Edidin [1] introduced the frame theoretic study of phaseless reconstruction. Since then, this has turned into a very active area of research. Over the years, many people have replaced the term *phaseless reconstruction* with *phase retrieval*. At a meeting in 2012, Casazza asked: *Are these really the same?* In this paper, we will show that phase retrieval is equivalent to phaseless reconstruction. We then show, more generally, that phase retrieval by projections is equivalent to phaseless reconstruction by projections in the real case.

1. Introduction

Throughout, $\mathbb{R}_m$ (resp. $\mathbb{C}_m$) will denote an m-dimensional real (resp. complex) Hilbert space, and $\mathbb{H}_m$ will denote an m-dimensional real or complex Hilbert space. Phase retrieval is an old problem in signal processing and has been studied for over 100 years by electrical engineers. We begin with a few necessary definitions and a brief history the problem before proceeding to the main result. Let $x = (a_1, a_2, \ldots, a_m)$ and $y = (b_1, b_2, \ldots, b_m)$ be vectors in $\mathbb{H}_m$. We say that x, y have the **same phase** if

$$\text{phase } a_i = \text{phase } b_i, \text{ for all } i = 1, 2, \ldots, m.$$

Definition 1.1. Given $\Phi = \{\phi_i\}_{i=1}^n$, a family of vectors in $\mathbb{H}_m$, let x and y be vectors in $\mathbb{H}_m$ such that

$$|\langle x, \phi_i\rangle| = |\langle y, \phi_i\rangle|, \text{ for all } i = 1, 2, \ldots, n. \tag{1}$$

i. We say Φ does **phase retrieval** if (1) implies there exists a $|\theta| = 1$ so that x and θy have the same phases.
ii. We say Φ does **phaseless reconstruction** if (1) implies there is a $|\theta| = 1$ so that $x = \theta y$.

☆ The authors were supported by NSF DMS 1307685; and NSF ATD 1321779; AFOSR: FA9550-11-1-0245.
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http://dx.doi.org/10.1016/j.jmaa.2015.11.045
0022-247X/

Moreover, in the real case, if $\theta = 1$ we say x and y have the **same signs** and if $\theta = -1$ we say x and y have **opposite signs**.

Note that from the definition of phase retrieval it is not immediately apparent that $x = \theta y$. The analogous definition for phase retrieval by projections follows.

Definition 1.2. Given $\{P_i\}_{i=1}^n$, a family of projections on $\mathbb{H}_m$, let x and y be vectors in $\mathbb{H}_m$ such that

$$\|P_i x\| = \|P_i y\|, \text{ for all } i = 1, 2, \ldots, n. \tag{2}$$

i. We say $\{P_i\}_{i=1}^n$ does **phase retrieval** if (2) implies there is a $|\theta| = 1$ so that x and θy have the same phases.
ii. We say $\{P_i\}_{i=1}^n$ does **phaseless reconstruction** if (2) implies there is a $|\theta| = 1$ so that $x = \theta y$.

In the setting of frame theory, the concept of phaseless reconstruction was introduced in 2006 by Balan, Casazza, and Edidin [1]. They showed that a *generic* family of $(2m-1)$-vectors in $\mathbb{R}_m$ does phaseless reconstruction, however no set of $(2m-2)$-vectors can. By *generic* we are referring to an open dense set in the set of $(2m-1)$-element frames in $\mathbb{H}_m$. In the complex case, they showed that a *generic* set of $(4m-2)$-vectors does phaseless reconstruction. Heinosaari, Mazzarella and Wolf improved this result in [7], where they showed that n-vectors doing phaseless reconstruction in $\mathbb{C}_m$ requires $n \geq 4m-4-2\alpha$, where α is the number of 1's in the binary expansion of $(m-1)$. This bound was further improved when Bodmann [3] showed that phaseless reconstruction in $\mathbb{C}_m$ can be done with $(4m-4)$-vectors. Later, Conca, Edidin, Hering, and Vinzant [5] showed that a *generic* frame with $(4m-4)$-vectors does phaseless reconstruction in $\mathbb{C}_m$. They also showed that if $m = 2^k + 1$ then no n-vectors with $n < 4m-4$ can do phaseless reconstruction. It was conjectured that no fewer than $(4m-4)$-vectors can do phaseless reconstruction. Recently, Vinzant [8] showed that this conjecture does not hold by producing 11 vectors in $\mathbb{C}_4$ which do phaseless reconstruction. The pursuit of improving this bound was finally resolved when Edidin [6] showed that a generic family of $2m-1$ subspaces will do phaseless in $\mathbb{R}^m$ and if $m = 2^k + 1$, then this bound is sharp. The analogous question for subspaces has still not been answered, however it was shown that there are six 2-dimensional subspaces of $\mathbb{R}^4$ which do phase retrieval (Xu [9]).

Over the years, we have replaced the phrase "*phaseless reconstruction*" with "*phase retrieval.*" Casazza, at a meeting in 2012, raised the question: "*Are these really the same?*" In this paper we will answer in the affirmative, by showing that in both the real and complex cases phase retrieval implies phaseless reconstruction (and similarly for phase retrieval by projections in $\mathbb{R}_m$).

This problem arose because of the way the engineering version of *phase retrieval* was translated into the language of frame theory. The engineers are working with the modulus of the Fourier transform and want to recover the phases to be able to invert the Fourier transform and recover the signal. In other words, the engineers only need is to recover the phase. But in the frame theory version of this, for $x = (a_1, a_2, \ldots, a_m)$, we are really trying to recover two things:

(1) Recover the phases of the a_i.
(2) Recover $|a_i|$ (which in the engineering case, is already known).

2. Phase retrieval versus phaseless reconstruction: real case

We begin by defining the *complement property* from [1].

Definition 2.1. A family of vectors $\{\phi_i\}_{i=1}^n$ in $\mathbb{H}_m$ has the **complement property** if for every $I \subset [n]$, either $\{\phi_i\}_{i\in I}$ spans $\mathbb{H}_m$ or $\{\phi_i\}_{i\in I^c}$ spans $\mathbb{H}_m$.