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A constructive method for plane-wave representations of special functions

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ABSTRACT

A general constructive scheme for the derivation of plane-wave representations of special functions is proposed. Illustrative examples of the construction are given. As one case study, new integral representations of the elliptic Weierstrass $\wp$ function are derived; these complement, and generalize, similar new plane-wave integral representations of the same function recently found by Dienstfrey and Huang (2006) [8] using other techniques. Our approach is inspired by recent developments in the so-called Fokas transform for the solution of boundary value problems for partial differential equations.

1. Introduction

The present work has been inspired by a recent paper of Dienstfrey & Huang [8] in which some new integral representations of the classical elliptic functions, and some related functions, are derived. Among other results, those authors derive the following integral representation of the Weierstrass $\wp$ function:

$$\wp(z) = \frac{1}{z^2} + 8\int_0^\infty \lambda \left[e^{-\lambda} \sinh^2\left(\frac{z\lambda}{2}\right) f_1(\lambda,\tau) + e^{i\tau\lambda} \sin^2\left(\frac{z\lambda}{2}\right) f_2(\lambda,\tau)\right] d\lambda, \tag{1}$$

where

$$f_1(\tau,\lambda) = \frac{\cosh^2(\tau\lambda/2)}{1 - 2e^{-\lambda}\cosh(\tau\lambda) + e^{-2\lambda}}, \qquad f_2(\tau,\lambda) = \frac{\cos^2(\lambda/2)}{1 - 2e^{i\tau\lambda}\cos(\lambda) + e^{2i\tau\lambda}}, \tag{2}$$

and τ is the lattice ratio. (1) is not valid for all $z \in \mathbb{C}$; it has a finite domain of validity as a consequence of the particular derivation given in [8]. Indeed, for (1) to be valid it is necessary that $z \in D(\tau)$ defined by

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$$D(\tau) = \{z|\mathrm{Re}[-1 \pm z \pm \tau] < 0, \mathrm{Im}(\tau \pm z) > 0\}. \tag{3}$$

Unfortunately, this region may not even include the entire fundamental polygon of the doubly periodic function. Dienstfrey & Huang [8] discuss the possibility of deriving fast numerical routines based on their new integral representations but they acknowledge the inconvenience of having such cumbersome restrictions on the domains of validity. Expression (1) is derived using Laplace–Mellin representations of multipoles together with results on the summation of two-dimensional geometric series.

The purpose of this paper is to find *other* – also apparently new – plane-wave integral representations for the Weierstrass $\wp$ function that have strong similarities to (1). But the results are different and, in particular, they enjoy the important distinction that they are valid throughout the entire fundamental polygon and with no restrictions on various sub-domains of validity. More significantly, the construction here is very general and we use the Weierstrass $\wp$ function as an example to showcase the possibilities of the method for other special functions.

Dienstfrey & Huang [8] were motivated in their enquiries by the fact that so-called "plane-wave representations" are a valuable tool in the development of recent embodiments of fast multipole methods [13,14]. In particular, an elementary but significant observation made by Hrycak & Rokhlin [14], and which underlies a significant improvement in the numerical implementation of fast multipole methods, is that if z and z' are complex numbers with $\mathrm{Re}(z - z') > 0$ then

$$\frac{1}{z - z'} = \int_0^\infty e^{-x(z-z')} dx. \tag{4}$$

This fact is stated as Lemma 2.8 of [14]. But precisely the *same* mathematical observation has recently been used by the author [4] as the basis for a new and elementary derivation of a transform pair for analytic functions defined over simply connected convex polygons of general shape. Those transform pairs were first written down by Fokas & Kapaev [12] who derived them using completely different methods based on the spectral analysis of a Lax pair and use of Riemann–Hilbert methods. The author's new derivation in [4] leads the way to extensions to find generalized transform pairs for *circular domains*, that is, domains whose boundaries are made up of circular arcs and straight lines [4,5] (this even includes multiply connected domains [5]).

Inspection of (1) reveals that its integrand is such that the only z dependence appears in purely exponential factors of the form $e^{\mathrm{i}\lambda z}$ (see (95) of §6 to see this explicitly). While referred to as "plane-wave representations" they are also known as *Ehrenpreis-type* integral representations [10] and the significance of the latter has resurfaced in recent years as a consequence of investigations into a new transform approach to both linear and nonlinear PDEs pioneered in recent decades by A.S. Fokas and collaborators [9,10,2].

One aim of the present paper is to indicate connections between all the mathematical ideas just described: plane wave representations, elliptic and other special functions, Ehrenpreis integral representations, and the Fokas transform method.

A key result here is to derive the following new plane-wave integral representation for the Weierstrass $\wp$ function with periods $2l$ and $2h\mathrm{i}$ for $l, h > 0$:

$$\wp(z) = \frac{1}{z^2} + \frac{1}{2\pi} \int_{L_1} \rho_1(k)[e^{\mathrm{i}kz} + e^{-\mathrm{i}kz} - 2]dk$$
$$+ \frac{1}{2\pi} \int_{L_2} \left[\rho_2(k) + \frac{k}{1 - e^{2\mathrm{i}kl}}\right] [e^{\mathrm{i}kz} + e^{-\mathrm{i}kz} - 2]dk, \tag{5}$$

where $\rho_1(k)$ and $\rho_2(k)$ are entire functions given explicitly by the rapidly convergent sums