J. Math. Anal. Appl. 436 (2016) 168–178

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

A Nicholson-type integral for the cross-product of the Bessel functions

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ARTICLE INFO

Article history:
Received 10 March 2015
Available online 4 December 2015
Submitted by K. Driver

Keywords:
Nicholson's formula
Cross-product
Bessel functions
Hankel functions

ABSTRACT

We prove a new Nicholson-type integral representation for the cross-product of the Bessel functions $J_\nu(w)Y_\nu(z) - Y_\nu(w)J_\nu(z)$, and related integral representations for $|H_\nu^{(1)}(z)|^2$, where $H_\nu^{(1)}$ is the Hankel function of the first kind.

1. Introduction

The main result of this paper is the following representation of the cross-product of the Bessel functions J_ν and Y_ν

$$J_\nu(w)Y_\nu(z) - Y_\nu(w)J_\nu(z) = \\ = \frac{2}{\pi} \int\limits_C J_0\left(\sqrt{z^2 + w^2 - 2zw\cosh\zeta}\right) \cosh\nu\zeta \, d\zeta, \tag{1}$$

where C is any contour starting at 0 and ending at $\log z - \log w$. The formula holds for every complex number ν, and all complex numbers z and w such that $|\operatorname{Arg} z| < \pi$ and $|\operatorname{Arg} w| < \pi$. We demonstrate it in Theorem 2.1 by verifying that the integral satisfies Bessel's differential equation and appropriate limiting conditions at the origin.

The cross-products of the Bessel functions have long been studied theoretically, see e.g. [13, 10.6, 10.21], [8], [7]. They have also found several applications to wave propagation in cylindrical geometries, e.g. to

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http://dx.doi.org/10.1016/j.jmaa.2015.11.056
0022-247X/

acoustic modeling of ducts [5], and to modeling of optical waveguides [6,14]. Other applications include heat transport [2] and free elastic oscillations of a spherical body [10]. Other integral representations of various products of the Bessel functions are well-known, an extensive list is given in [9, pp. 93–98], see also [11].

Our second result is an integral representation of the square of the modulus of the Hankel function $H_\nu^{(1)}$ of the first kind

$$|H_\nu^{(1)}(x+\imath y)|^2 = \frac{8}{\pi^2}\int_0^\infty K_0\left(2\sqrt{x^2\sinh^2 t + y^2\cosh^2 t}\right)\cosh 2\nu t\, dt. \tag{2}$$

This formula is valid if all x, y, ν are real, and $y > 0$. We present two simple derivations: one in Theorem 3.1 is based on a known representation of the product of the modified Bessel functions $K_\nu(\alpha)K_\nu(\beta)$, and another one in Appendix uses the Mehler–Sonine integrals.

Formula (2) has some interesting consequences. For example, it follows from (2) that for a fixed z in the upper half-plane, the magnitude $|H_\nu^{(1)}(z)|$ increases with the order ν when $\nu \geqslant 0$. This fact has been observed experimentally [4,1], but no proofs are given there. The property is needed in the numerical evaluation of the Bessel and the Hankel functions using a three-term recurrence. For the sake of numerical stability, the recurrence must proceed in the direction of increasing magnitudes. It is known that the magnitudes of the modified Bessel function $|K_\nu(\zeta)|$ grow with $\nu \geqslant 0$ [13, 10.37] when ζ is in the right half-plane, and monotonicity of $|H_\nu^{(1)}(z)|$ can also be deduced from this fact.

In 1910, J.W. Nicholson published [12] the following equality

$$J_\nu^2(z) + Y_\nu^2(z) = \frac{8}{\pi^2}\int_0^\infty K_0(2z\sinh t)\,\cosh 2\nu t\; dt, \tag{3}$$

valid for $\mathrm{Re}(z) > 0$ and an arbitrary complex order ν. Nicholson's formula can be recovered from (2) by fixing $x > 0$, letting $y \to 0^+$, and then taking analytic continuation first in z, and then in ν. This derivation of (3) appears somewhat simpler than the one presented in [15], while another short proof is given in [16].

Finally, in Theorem 3.2 we combine (1) and (2) to obtain the following formula

$$\begin{aligned}|H_\nu^{(1)}(x-\imath y)|^2 = {}& \frac{8}{\pi^2}\int_0^\infty K_0\left(2\sqrt{x^2\sinh^2 t + y^2\cosh^2 t}\right)\cosh 2\nu t\; dt\\ &+ \frac{8}{\pi}\int_0^{\operatorname{arccot}\frac{x}{y}} I_0\left(2\sqrt{-x^2\sin^2\phi + y^2\cos^2\phi}\right)\cos 2\nu\phi\; d\phi,\end{aligned}$$

which is valid if all x, y, ν are real, and $y > 0$.

2. Integral representation of the cross-product

The following theorem, which gives an integral representation of the cross-product of the Bessel functions J_ν and Y_ν, is the main result of this paper.

Theorem 2.1. *For all complex numbers z and w such that $|\operatorname{Arg} z| < \pi$ and $|\operatorname{Arg} w| < \pi$, and for every complex number ν,*

$$J_\nu(w)Y_\nu(z) - Y_\nu(w)J_\nu(z) = \tag{4}$$

$$= \frac{2}{\pi}\int_C J_0\left(\sqrt{z^2+w^2-2zw\cosh\zeta}\right)\,\cosh\nu\zeta\; d\zeta, \tag{5}$$

where C is any contour starting at 0 and ending at $\log z - \log w$.