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Uniform estimates for the flow of a viscous incompressible fluid down an inclined plane in the thin film regime

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A R T I C L E   I N F O

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A B S T R A C T

We consider a two-dimensional motion of a thin film flowing down an inclined plane under the influence of the gravity and the surface tension. In order to investigate the stability of such flow, it is hard to treat the Navier–Stokes equations directly, so that a thin film approximation is often used. It is an approximation obtained by the perturbation expansion with respect to the aspect ratio δ of the film under the thin film regime $\delta \ll 1$. Our purpose is to give a mathematically rigorous justification of the thin film approximation by establishing an error estimate between the solution of the Navier–Stokes equations and those of approximate equations. To this end, in this paper we derive a uniform estimate for the solution of the Navier–Stokes equations with respect to δ under appropriate assumptions.

1. Introduction

In this paper, we consider a two-dimensional motion of a liquid film of a viscous and incompressible fluid flowing down an inclined plane under the influence of the gravity and the surface tension on the interface. The motion can be mathematically formulated as a free boundary problem for the incompressible Navier–Stokes equations. We assume that the domain $\Omega(t)$ occupied by the liquid at time $t \geq 0$, the liquid surface $\Gamma(t)$, and the rigid plane Σ are of the forms

$$\begin{cases} \Omega(t) = \{(x,y) \in \mathbb{R}^2 \mid 0 < y < h_0 + \eta(x,t)\}, \\ \Gamma(t) = \{(x,y) \in \mathbb{R}^2 \mid y = h_0 + \eta(x,t)\}, \\ \Sigma = \{(x,y) \in \mathbb{R}^2 \mid y = 0\}, \end{cases}$$

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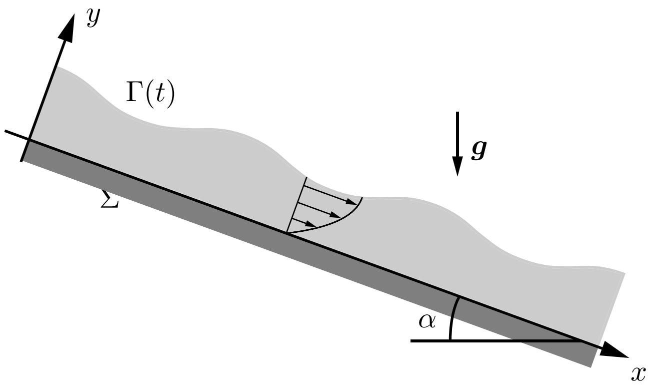

Fig. 1. Sketch of a thin liquid film flowing down an inclined plane.

where h_0 is the mean thickness of the liquid film and $\eta(x,t)$ is the amplitude of the liquid surface. Here we choose a coordinate system (x,y) so that x axis is down and y axis is normal to the plane (see Fig. 1). The motion of the liquid is described by the velocity $\boldsymbol{u} = (u,v)^{\mathrm{T}}$ and the pressure p satisfying the Navier–Stokes equations

$$\begin{cases} \rho\big(\boldsymbol{u}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\big) = \nabla\cdot\mathbf{P} + \rho g(\sin\alpha, -\cos\alpha)^{\mathrm{T}} & \text{in} \quad \Omega(t),\ t>0, \\ \nabla\cdot\boldsymbol{u} = 0 & \text{in} \quad \Omega(t),\ t>0, \end{cases} \tag{1.1}$$

where $\mathbf{P} = -p\mathbf{I} + 2\mu\mathbf{D}$ is the stress tensor, $\mathbf{D} = \frac{1}{2}\big(D\boldsymbol{u} + (D\boldsymbol{u})^{\mathrm{T}}\big)$ is the deformation tensor, $\mathbf{I}$ is the unit matrix, ρ is a constant density of the liquid, g is the acceleration of the gravity, α is the angle of inclination, and μ is the shear viscosity coefficient. The dynamical and kinematic conditions on the liquid surface are

$$\begin{cases} \mathbf{P}\boldsymbol{n} = -p_0\boldsymbol{n} + \sigma H\boldsymbol{n} & \text{on} \quad \Gamma(t),\ t>0, \\ \eta_t + u\eta_x - v = 0 & \text{on} \quad \Gamma(t),\ t>0, \end{cases} \tag{1.2}$$

where $\boldsymbol{n}$ is the unit outward normal vector to the liquid surface, that is, $\boldsymbol{n} = \frac{1}{\sqrt{1+\eta_x}}(-\eta_x, 1)^{\mathrm{T}}$, p_0 is a constant atmospheric pressure, σ is the surface tension coefficient, and H is the twice mean curvature of the liquid surface, that is, $H = \Big(\frac{\eta_x}{\sqrt{1+\eta_x^2}}\Big)_x$. The boundary condition on the rigid plane is the non-slip condition

$$\boldsymbol{u} = \mathbf{0} \quad \text{on} \quad \Sigma,\ t>0. \tag{1.3}$$

These equations have a laminar steady solution of the form

$$\eta = 0, \quad u = (\rho g\sin\alpha/2\mu)(2h_0 y - y^2), \quad v = 0, \quad p = p_0 - \rho g\cos\alpha(y - h_0), \tag{1.4}$$

which is called the Nusselt flat film solution. Throughout this paper, we assume that the flow is downward l_0-periodic or approaches asymptotically this flat film solution at spatial infinity.

Concerning the instability of this laminar flow, there are vast research literatures from the physical and engineering point of view. The first investigation of the wave motion of thin film including the effect of the surface tension was provided by Kapitza [11]. In particular, he considered the case where the liquid film flows down a vertical wall, that is, the case $\alpha = \frac{\pi}{2}$. Yih [23] first formulated the linear stability problem of the laminar flow of the liquid film flowing down an inclined plane as an eigenvalue problem for the complex phase velocity, more specifically, the Orr–Sommerfeld problem although he neglected the effect of the surface tension. Benjamin [3] took into account the effect of the surface tension and showed that the critical Reynolds number is given by $\mathrm{R}_c = \frac{5}{4}\frac{1}{\tan\alpha}$ by expanding the normal mode solution in powers