J. Math. Anal. Appl. 436 (2016) 305–321

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Asymptotic decomposition of substochastic operators and semigroups

Katarzyna Pichór [a], Ryszard Rudnicki [b,*]

[a] Institute of Mathematics, Silesian University, Bankowa 14, 40-007 Katowice, Poland
[b] Institute of Mathematics, Polish Academy of Sciences, Bankowa 14, 40-007 Katowice, Poland

ARTICLE INFO

Article history:
Received 5 June 2015
Available online 11 December 2015
Submitted by U. Stadtmueller

Keywords:
Stochastic operators
Semigroups
Asymptotic stability
Asymptotic decomposition

ABSTRACT

Theorems on asymptotic decomposition of substochastic operators and semigroups are given. General results are applied to show asymptotic periodicity, asymptotic stability and sweeping of substochastic operators and semigroups.

1. Introduction

Substochastic operators and semigroups appear in many fields of mathematics and applications. They are used to study ergodic properties of dynamical systems [13], the evolution of Markov chains [4,15,20], and the long-time behaviour of Markov processes like diffusion processes, piecewise deterministic Markov processes (PDMPs) and hybrid stochastic processes [23]. The key role in investigations of substochastic operators and semigroups plays their long-time behaviour: asymptotic stability (strong mixing) [13], sweeping [12] (zero type, null property [15]) or more subtle as completely mixing [14], zero-two law [5] or convergence after rescaling (e.g. the central limit theorems). Although the study of asymptotic properties of substochastic operators and semigroups has a long history and the core of the theory was formulated in the late sixties [4,8–10], new interesting theoretical results still appear, e.g. [2,7,27].

Nowadays the development of the theory is also stimulated by applications in natural sciences, where substochastic semigroups generated by degenerated diffusion [16,22,24], PDMPs [1,3,19,25] or stochastic hybrid systems [18,23] play special role. The main problem in application of the existing theoretical results concerning long-time behaviour of substochastic operators or semigroups is that most of them based on *a priori* assumptions concerning their asymptotic behaviour, e.g., irreducibility, recurrence, transience,

* Corresponding author.
E-mail addresses: katarzyna.pichor@us.edu.pl (K. Pichór), rudnicki@us.edu.pl (R. Rudnicki).

http://dx.doi.org/10.1016/j.jmaa.2015.12.009
0022-247X/

conservativity, dissipativeness, or overlaping supports. The aim of our paper is to give some general results concerning asymptotic decomposition of substochastic operators and semigroups, which do not base on such assumptions. We consider only operators and semigroups which have non-trivial integral parts, i.e. they are *partially integral* (*partially kernel*), often appearing in PDMPs or stochastic hybrid systems. We consider substochastic operators on the space $L^1(X)$, where we assume that X is a separable metric space. The main idea is to give a condition on the kernel part of the semigroup (see condition (K)) which is easy to verify and which allows us to represent an asymptotic decomposition of an operator or a semigroup in the terms of strong asymptotic convergence or sweeping from compact sets. A similar decomposition result for semigroups enjoying the strong Feller property and being weakly ergodic was proved in [6]. Condition (K) was proposed in [21] as a tool to study sweeping of stochastic operators. In cases of substochastic semigroups generated by PDMPs or degenerated diffusions condition (K) can be checked by finding the dimension of some space spanned by vectors used in their descriptions (a kind of the Hörmander condition) see [19,23,24]. Condition (K) is of local type what makes it different from the *minorization condition* $M(m_0, 1, s, \nu)$ usually used to prove convergence to a positive distribution [15].

The main idea of the proofs of our theorems is to reduce the problem to be treated by the well known theory of Harris operators. The problem is that our operator (or semigroup) can be neither conservative nor dissipative, we even do not know if it is indecomposable. It should be underlined that both assumptions and theses of our theorems use topological notions and it will be difficult to formulate these results only in terms of a measure space. Our theorems generalized some earlier results [17,21], but also unify a part of the theory of substochastic operators and semigroups related to asymptotic stability and sweeping.

The organization of the paper is as follows. In Section 2 we collect relevant definitions and formulate the main results (Theorems 1 and 2) concerning asymptotic decomposition of substochastic operators and semigroups. From these theorems we deduce corollaries concerning sweeping from compact sets. It should be noted that we do not assume here that the operator (or semigroup) is irreducible as in [21]. We also formulate some well known results (Propositions 1 and 2) concerning asymptotic periodicity of operators and asymptotic stability of semigroups which do not follow directly from Theorems 1 and 2 but can be easily obtained from lemmas used in the proofs of theorems. Section 3 contains some auxiliary definitions and results concerning partially integral operators and the proofs of results formulated in Section 2. In the last section we present some corollaries in the case when the operator (semigroup) is additionally *irreducible*. The assumption of irreducibility allows us to express our results in the form of the Foguel alternative [11,13], i.e., the stochastic semigroup is asymptotically stable or sweeping from compact sets. We also give some counterexamples which show the necessity of condition (K) in Theorems 1 and 2 and the assumption that the semigroup is partially integral in Proposition 2.

2. Formulation of the results

Let the triple (X, Σ, m) be a σ-finite measure space. A linear operator $P\colon L^1(X, \Sigma, m) \to L^1(X, \Sigma, m)$ is called *substochastic* if it is positive, i.e. if $f \geq 0$ then $Pf \geq 0$ for $f \in L^1(X, \Sigma, m)$, and if $\|Pf\| \leq \|f\|$ for $f \in L^1(X, \Sigma, m)$. We denote by $P^*\colon L^\infty(X, \Sigma, m) \to L^\infty(X, \Sigma, m)$ the adjoint operator of P. In the whole paper we will consider substochastic operators which have transition functions. We recall that $\mathcal{P}(x, A)$ is a transition function on (X, Σ) if for each $x \in X$ the function $A \mapsto \mathcal{P}(x, A)$ is a measure on (X, Σ) such that $\mathcal{P}(x, X) \leq 1$ and the function $x \mapsto \mathcal{P}(x, A)$ is measurable for each $A \in \Sigma$. The transition function $\mathcal{P}$ corresponds to a substochastic operator P if

$$P^*g(x) = \int g(y)\, \mathcal{P}(x, dy) \quad \text{for } g \in L^\infty(X, \Sigma, m). \tag{1}$$