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On exceptional sets in Erdős–Rényi limit theorem

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ABSTRACT

For $x \in [0,1]$, the run-length function $r_n(x)$ is defined as the length of the longest run of 1's amongst the first n dyadic digits in the dyadic expansion of x. Erdős and Rényi proved that $\lim_{n\to\infty} \frac{r_n(x)}{\log_2 n} = 1$ for Lebesgue almost all $x \in [0,1]$. In this paper, we study the Hausdorff dimensions of the exceptional sets in Erdős–Rényi limit theorem. Let $\varphi : \mathbb{N} \to (0, +\infty)$ be a monotonically increasing function satisfying $\lim_{n\to\infty} \frac{n}{\varphi(n^{1+\alpha})} = +\infty$ with some $0 < \alpha \le 1$. We prove that the set

$$E_{\max}^{\varphi} = \left\{ x \in [0,1] : \liminf_{n\to\infty} \frac{r_n(x)}{\varphi(n)} = 0, \limsup_{n\to\infty} \frac{r_n(x)}{\varphi(n)} = +\infty \right\}$$

has Hausdorff dimension one and is residual in $[0,1]$.

1. Introduction

Let us begin with the definition of run-length function $r_n(x)$, which was introduced to measure the length of consecutive terms of "heads" in n Bernoulli trials. More precisely, recall that any $x \in [0,1]$ can be represented as

$$x = \sum_{k=1}^{\infty} \frac{x_k}{2^k},$$

where $x_k \in \{0,1\}$ for any $k \ge 1$. Write $\Sigma = \{0,1\}^{\mathbb{N}}$. The infinite sequence $(x_1, x_2, x_3, \ldots) \in \Sigma$ is called the digits sequence of x. Let $\pi : \Sigma \to [0,1]$ be the code map, that is, $\pi((x_1, x_2, x_3, \ldots)) = x$. For each $n \ge 1$ and $x \in [0,1]$, the run-length function $r_n(x)$ is defined as the length of the longest run of 1's in $(x_1, x_2, \ldots, x_n)$,

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that is,

$$r_n(x) = \max\{\ell : x_{i+1} = \cdots = x_{i+\ell} = 1 \text{ for some } 0 \le i \le n - \ell\}.$$

The run-length function has been extensively studied in probability theory and used in reliability theory, biology, quality control. For the asymptotic behavior of r_n, Erdős and Rényi [6] (see also [23]) proved that, for Lebesgue almost all $x \in [0, 1]$,

$$\lim_{n\to\infty} \frac{r_n(x)}{\log_2 n} = 1.$$

That is, the rate of growth of $r_n(x)$ is $\log_2 n$ for almost all $x \in [0, 1]$. Zou [24] considered some special sets consisting of points whose run-length functions obey other asymptotic behavior instead of $\log_2 n$. Chen and Wen [5] studied some level sets involved in the frequency on integer expansion and run-length function. For more details about the run-length function, we refer the reader to the book [23].

It is natural to study the exceptional set in the above Erdős–Rényi limit theorem. Ma et al. [16] proved that the set of points that violate the above Erdős and Rényi law is visible in the sense that it has full Hausdorff dimension.

Let

$$E = \left\{x \in [0, 1] : \liminf_{n\to\infty} \frac{r_n(x)}{\log_2 n} < \limsup_{n\to\infty} \frac{r_n(x)}{\log_2 n}\right\}.$$

It is worth to point out that E is smaller than the set of points that violate the above Erdős and Rényi law because we consider the asymptotic behavior of $r_n(x)$ with respect to the fixed speed $\log_2 n$. It follows from the Erdős–Rényi limit theorem that the set E is negligible from the measure-theoretical point of view. There is a natural question: what is the Hausdorff dimension of the set E? In fact, questions related to the exceptional sets from dynamics and fractals have recently attracted huge interest in the literature. Generally speaking, exceptional sets are big from the dimensional point of view, and they have the same fractal dimensions as the underlying phase spaces, see [1,4,8,9,11,14,15,18,20–22] and references therein. In this paper we study a class of extremely refined subsets of the set E. Define

$$E_{\max} = \left\{x \in [0, 1] : \liminf_{n\to\infty} \frac{r_n(x)}{\log_2 n} = 0, \limsup_{n\to\infty} \frac{r_n(x)}{\log_2 n} = +\infty\right\}. \tag{1}$$

That is, $E_{\max}$ is the set consisting of those "worst" divergence points. Clearly, $E_{\max} \subset E$. We are interested in the Hausdorff dimension of the set $E_{\max}$.

Intuitively, we feel that the set $E_{\max}$ shall be "small". However, we have the following somewhat surprising result.

Theorem 1.1. *Let $E_{\max}$ be defined as in (1). Then*

$$\dim_H E_{\max} = 1.$$

Here and in the sequel, $\dim_H E$ denotes the Hausdorff dimension of the set E. For more details about Hausdorff dimension and the theory of fractal dimensions, we refer the reader to the famous book [7].

It is also natural to study the asymptotic behavior of run-length function with respect to other speeds instead of $\log_2 n$. Next we will show that Theorem 1.1 still holds for a general class of sets. More precisely, let $\varphi : \mathbb{N} \to (0, +\infty)$ be a monotonically increasing function. Define