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Non-power bounded generators of strongly continuous semigroups

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ABSTRACT

It is folklore that a power bounded operator on a sequentially complete locally convex space generates a uniformly continuous C_0-semigroup which is given by the corresponding power series representation. Recently, Domański asked if in this result the assumption of being power bounded can be relaxed. We employ conditions introduced by Żelazko to give a weaker but still sufficient condition for generation and apply our results to operators on classical function and sequence spaces.

1. Introduction

Given a Banach space, then every linear and continuous operator from the space into itself generates a C_0-semigroup which is given by an exponential series representation. In the Banach space world of C_0-semigroups this case of a continuous generator is thus considered to be the trivial situation. The picture changes completely already in the considerably harmless appearing case of complete metrizable spaces. Based on a question of Conejero [7] several relations between continuity of the generator, uniform continuity of the semigroup and validity of a series representation have recently been revealed by Albanese, Bonet, Ricker [1, Thm. 3.3 and Prop. 3.2] and Frerick, Jordá, Kalmes, Wengenroth [10].

In the general case of a sequentially complete locally convex space X, it seems that the only result available so far is the generation theorem mentioned in the book [14] of Yosida, which states that a power bounded operator is always a generator. Here, $A \in L(X)$ is *power bounded* if all its powers form an equicontinuous subset of $L(X)$, i.e., if for every continuous seminorm p on X there exists a continuous seminorm q on X and a constant $M > 0$ such that the estimate $p(A^n x) \leqslant Mq(x)$ holds for all $n \in \mathbb{N}_0$ and all $x \in X$. It is straightforward to generalize the above statement as follows.

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Theorem 0. *Let $A \in L(X)$. Assume that there exists $\mu > 0$ such that μA is power bounded. Then A generates a uniformly continuous C_0-semigroup $(T(t))_{t\geqslant 0}$ which is given by the formula $T(t) = \sum_{n=0}^{\infty}(tA)^n/n!$ for $t \geqslant 0$ where the series converges absolutely with respect to the topology of uniform convergence on the bounded subsets of X.*

Inspired by Allan [2, p. 400] in the current paper operators with the property assumed in Theorem 0 are said to be *a-bounded*. Recently, Domański [8] asked, if the statement above can be improved in the sense of a condition weaker than a-boundedness which still assures the series representation or at least the generator property. Frerick, Jordá, Kalmes, Wengenroth [10] characterized generation for a special class of Fréchet spaces by a condition closely related to the notions of *m-topologizable* and *topologizable* operators due to Żelazko [15,16].

In this paper we consider a quantitative version of topologizability which is weaker than m-topologizability but implies the generator property and guarantees a series representation. Also for the case of an m-topologizable operator this result seems to be new and improves Theorem 0. In combination with Bonet [4, Ex. 6] the latter shows that there exist complete, non-normable spaces on which every continuous operator is a generator—as in the case of a Banach space—although in general this is well-known to be not the case, see [10, Ex. 1]. We provide examples which show that our result applies to a class of operators which is strictly larger than those of the m-topologizable ones. Our counterexamples show that topologizability alone in general neither is necessary nor sufficient for generation. A variation of our main result, see Theorem 2, suggests that it might be possible that an operator generates a C_0-semigroup but that the series representation is only valid on a finite time interval and fails for large times. It is open if such C_0-semigroups do really occur in nature.

For the theory of locally convex spaces we refer to Meise, Vogt [13] and Jarchow [11]. For basic facts about semigroups on locally convex spaces we refer to Yosida [14] and Kōmura [12, Section 1].

2. Notation

For the whole article let X be a sequentially complete locally convex space. We denote by $\operatorname{cs}(X)$ the system of all continuous seminorms on X, by $\mathcal{B}$ the collection of all bounded subsets of X and by $L(X)$ the space of all linear and continuous maps from X into itself. We write $L_b(X)$, if $L(X)$ is furnished with the topology of uniform convergence on the bounded subsets of X given by the seminorms

$$q_B(S) = \sup_{x\in B} q(Sx)$$

for $S \in L(X)$, $B \in \mathcal{B}$ and $q \in \operatorname{cs}(X)$. Under a C_0-semigroup $(T(t))_{t\geqslant 0}$ on X we understand a family of maps $T(t) \in L(X)$ such that $T(0) = \mathrm{id}_X$, $T(t+s) = T(t)T(s)$ for t, $s \geqslant 0$ and $\lim_{t\to t_0} T(t)x = T(t_0)x$ for $x \in X$ and $t_0 \geqslant 0$. $(T(t))_{t\geqslant 0}$ is said to be uniformly continuous if $T(\cdot)\colon [0,\infty) \to L_b(X)$ is continuous. The generator $A\colon D(A) \to X$ of a C_0-semigroup $(T(t))_{t\geqslant 0}$ is defined by

$$Ax = \lim_{t\searrow 0} \frac{T(t)x - x}{t} \quad \text{for } x \in D(A) = \Big\{x \in X\,;\, \lim_{t\searrow 0} \frac{T(t)x - x}{t} \text{ exists}\Big\}.$$

The aim of this article is to identify conditions which guarantee that for a given $A \in L(X)$ there exists a C_0-semigroup $(T(t))_{t\geqslant 0}$ such that A is its generator. In Section 1 we mentioned already a classical condition of this type. In the remainder we use the following three conditions which appeared in the literature in different contexts. The first condition below is the assumption of Theorem 0. In view of Allan [2, p. 400] we say that an operator is *a-bounded*, if

$$\exists\, \mu > 0\; \forall\, p \in \operatorname{cs}(X)\; \exists\, q \in \operatorname{cs}(X)\; \forall\, n \in \mathbb{N}_0,\, x \in X\colon p(A^n x) \leqslant \mu^n q(x) \tag{1}$$