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Four small limit cycles around a Hopf singular point in 3-dimensional competitive Lotka–Volterra systems

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A B S T R A C T

The 3-dimensional competitive Lotka–Volterra (LV) systems have been studied for more than two decades, and particular attention has been focused on bifurcation of limit cycles. For such a system, Zeeman (1993) identified 33 stable equivalence classes on a carrying simplex, among which only classes 26–31 may have limit cycles. It has been shown that all these 6 classes may possess two limit cycles, and the existence of three limit cycles was claimed in some of these classes. Recently, Gyllenberg and Yan (2009) studied the existence of four limit cycles, three of them are small-amplitude limit cycles due to Hopf bifurcation and one additional limit cycle, enclosing all the three small-amplitude limit cycles, is due to the existence of a heteroclinic cycle, and proposed a new conjecture including: (i) There exists a 3-d competitive LV system with at least 5 limit cycles. (ii) In the case of a heteroclinic cycle on the boundary of the carrying simplex of a 3-d competitive LV system, the vanishing of the first four focus values (the vanishing of the zero-order focus value means that there is a pair of purely imaginary eigenvalues at the positive equilibrium) does not imply that the heteroclinic cycle is neutrally stable, and hence it does not imply that the positive equilibrium is a center. (iii) In the case of a heteroclinic cycle on the boundary of the carrying simplex of a 3-d competitive LV system, the vanishing of the first three focus values and that the heteroclinic cycle is neutrally stable do not imply the vanishing of the third-order focus value, and hence they do not imply that the positive equilibrium is a center. In this paper, we will present two examples belonging to class 27 and another two examples belonging to class 26, which exhibit at least four small-amplitude limit cycles in the vicinity of the positive equilibrium due to Hopf bifurcations, and prove that the items (ii) and (iii) in the conjecture are true. Moreover, showing the existence of four small-amplitude limit cycles is a necessary step towards proving item (i) of the conjecture.

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1. Introduction

Limit cycle is an important phenomenon in dynamical systems, which can occur in almost all fields of science and engineering such as physics, mechanics, electronics, ecology, economy, biology, finance etc. Analysis of limit cycles plays an important role in the study of nonlinear systems since bifurcation of limit cycles can cause complex dynamics in such systems. In recent years, bifurcation of limit cycles has been investigated in many biological systems (e.g. see [2,15]). One well-known example is the 3-dimensional Lotka–Volterra (LV) system. It is known that 2-dimensional LV systems cannot have limit cycles [1]. While for general 3-dimensional LV systems, complicated dynamical behavior such as period doubling route to chaos has been found (e.g., see [6]). On the other hand, for 3-dimensional competitive LV systems, described by

$$\dot{z}_i = z_i\Big(r_i - \sum_{j=1}^{3} a_{ij} z_j\Big), \quad i = 1, 2, 3, \quad r_i > 0, \quad a_{ij} > 0, \tag{1.1}$$

the dynamical possibilities are more restricted. Here, r_i and a_{ij} are constant parameters. In general, the system can have one positive equilibrium and seven boundary equilibria. Hirsch [14] first showed that all non-trivial orbits of system (1.1) approach an invariant two-dimensional manifold, called "carrying simplex", leading to a Poincaré–Bendixson theorem to be held for system (1.1). Thus, the long-term behavior of system (1.1) is determined by the dynamics on this simplex. Later, Zeeman [28] identified 33 stable equivalence classes in system (1.1), and showed that in 27 of these classes, all the compact limit sets are fixed points, so the dynamical behavior for these cases have been fully described. Further, Hopf bifurcation theory was applied to show that the remaining 6 classes (26–31) can possess isolated periodic orbits or limit cycles, and only one class (class 27) may have heteroclinic cycle. Since then, the open question of how many limit cycles can surround the positive equilibrium has attracted many researchers to investigate. Some results were obtained and a couple of conjectures were posed on the maximal number of limit cycles.

Twenty years ago, Hofbauer and So [16] showed two limit cycles for class 27, one of which is generated by a Hopf bifurcation and the other is guaranteed by the Poincaré–Bendixson theorem due to the existence of a heteroclinic cycle. They proposed a conjecture as described below.

Conjecture 1.1. *(See [16].) For system (1.1), in the case of heteroclinic cycle on the boundary of the carrying simplex, the following conditions are equivalent to having a center:*

(a) There is a pair of purely imaginary eigenvalues at the positive equilibrium.
(b) The first focus value vanishes.
(c) The heteroclinic cycle is neutrally stable.
And further, condition (c) might be replaced by the condition,
(c') The second focus value vanishes.

Thus, according to this conjecture, the maximal number of the limit cycles that system (1.1) can have might be two.

Later, Xiao and Li [24] proved that the number of limit cycles bifurcating in system (1.1) is finite if the system does not have heteroclinic cycles. Moreover, in this paper, an example is also given to show the existence of two limit cycles for class 27. Similar to the example given in [16] these two limit cycles contain one limit cycle due to a Hopf bifurcation and the other due to the existence of a heteroclinic cycle. Further, two limit cycles were found in [19] for classes 26, 28, 29, and in [9] for classes 30, 31. In 2003, Lu and Luo claimed that they obtained three limit cycles for class 27 [20]. Three years later, three limit cycles were also found for class 29 [11]. Recently, Gyllenberg and Yan constructed examples for class 27 and claimed that