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Normal structure and invariance of Chebyshev center under isometries

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A R T I C L E   I N F O

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A B S T R A C T

A normed space is said to have the weak fixed point property (WFPP) if every nonexpansive self map on a weakly compact convex set has a fixed point. Kirk proved that if a normed space has normal structure, then it has WFPP. It is known that if a normed space is uniformly convex in every direction (UCED), then it has normal structure. Also known is that every normed space that is uniformly convex in all but countably many directions has normal structure. We show that a normed space X has normal structure if the set of all directions in which it is not uniformly convex is contained in a countable union of n-dimensional subspaces of X for some positive integer n. We also show that in such a space, the Chebyshev center $C(K)$ of a weakly compact convex set K is a common invariant set for the collection of all isometries that map K into K and also that there is a common fixed point in $C(K)$ for this collection of maps. This was previously known to be true only in the case of a normed space that is UCED. Another observation made in this paper is that a Banach space X has normal structure if the set of all directions in which it is not uniformly convex is contained in a linear subspace with a countable Hamel basis.

1. Introduction

The concept of normal structure was introduced by Brodskii and Milman in [2] in order to study fixed points of isometries. Subsequently, Kirk proved in [9] that if a weakly compact convex set of a normed space has normal structure, then it has the fixed point property for nonexpansive maps. Since then, a great deal of study [1,3–6,10,14,15,17,18] has been made in connection with identifying normed spaces that have normal structure and its generalizations. Lim proved in [11] that if a weakly compact convex set has normal structure then any commuting family of nonexpansive selfmaps on the set has a common fixed point.

In [21], Zizler proved that a normed space that is uniformly convex in every direction (UCED) has normal structure. Later, Smith showed in [20] that a normed space that is uniformly convex in all but countably many directions has normal structure. In this paper we consider the case in which a normed space is

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uniformly convex in all directions except those that are contained in a countable union of n-dimensional subspaces, for some positive integer n. In Section 3, we show that a normed space with this property has normal structure. Besides, we show that a Banach space has normal structure if the set of directions in which it is not uniformly convex is contained in a linear subspace with a countable Hamel Basis. We show by a simple example that this need not be true for a normed space that is not complete.

In [2], it is proved that if K is a weakly compact convex subset of a normed space and K has normal structure, then K contains a point which is fixed by every isometry that maps K onto K. From [13], we know that in a normed space that is UCED, every weakly compact convex subset K contains a point which is fixed by every isometry from K into K. In Section 4, we show that this happens in any normed space that is uniformly convex in all directions except those contained in a countable union of n-dimensional subspaces for some positive integer n. In Section 5, we give an example of such a space.

2. Preliminaries

Given a normed space X, let S_X denote the unit sphere of X. For a subset A of X, let $\mathrm{co}(A) = \{\lambda_1 x_1 + \cdots + \lambda_m x_m : m \in \mathbb{Z}^+,\ x_1, \ldots, x_m \in A,\ \lambda_1, \ldots, \lambda_m \geq 0 \text{ and } \lambda_1 + \cdots + \lambda_m = 1\}$ and $\mathrm{aff}(A) = \{\lambda_1 x_1 + \cdots + \lambda_m x_m : m \in \mathbb{Z}^+,\ x_1, \ldots, x_m \in A,\ \lambda_1, \ldots, \lambda_m \in \mathbb{R} \text{ and } \lambda_1 + \cdots + \lambda_n = 1\}$. The sets $\mathrm{co}(A)$ and $\mathrm{aff}(A)$ are called the convex hull and the affine hull of A respectively. Their closures are denoted by $\overline{\mathrm{co}}(A)$ and $\overline{\mathrm{aff}}(A)$ respectively.

Definition 2.1. (See Garkavi [7].) For each $\epsilon \in [0, 2]$ and $z \in S_X$, let

$$\delta_z(\epsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} : x, y \in S_X, \|x - y\| \geq \epsilon, x - y = \alpha z, \alpha \text{ is a scalar} \right\}.$$

A normed space X is said to be uniformly convex (uniformly rotund) in the direction of $z \in S_X$ if for every $\epsilon \in (0, 2]$, $\delta_z(\epsilon)$ is positive. A normed space is said to be uniformly convex in every direction (UCED) if it is uniformly convex in the direction of each $z \in S_X$.

The function δ_z is called the modulus of convexity (rotundity) [19] of X in the direction of z, for each $z \in S_X$.

It is known that a normed space is uniformly convex in the direction of $z \in S_X$ if and only if

$$\inf \left\{ 1 - \frac{\|x + y\|}{2} : x, y \in B_X, \|x - y\| \geq \epsilon, x - y = \alpha z, \alpha \text{ is a scalar} \right\} > 0$$

for every $\epsilon \in (0, 2]$.

This is an easy consequence of the following result from [7].

Proposition 2.1 *(Garkavi). Let X be a normed space and let $z \in S_X$. If there exist sequences (a_m) and (b_m) in X such that $\lim_{m\to\infty} \|a_m\| = 1 = \lim_{m\to\infty} \|b_m\|$, $a_m - b_m \in span\{z\}$, $\lim_{m\to\infty} \left\| \frac{a_m + b_m}{2} \right\| = 1$ and $\|a_m - b_m\|$ does not tend to 0, then X is not uniformly convex in the direction of z.*

Given a normed space X and a nonempty bounded subset K of X we let $diam(K) = \sup\{\|x - y\| : x, y \in K\}$, $R(x, K) = \sup\{\|x - y\| : y \in K\}$ for all $x \in X$, $R(K) = \inf\{R(x, K) : x \in K\}$ and $C(K) = \{x \in K : R(x, K) = R(K)\}$. We call $diam(K)$, $R(K)$ and $C(K)$, the diameter, the Chebyshev radius and the Chebyshev center of K respectively. We say a point $x \in K$ is a diametral point of K if $R(x, K) = diam(K)$ and nondiametral otherwise.