J. Math. Anal. Appl. 434 (2016) 1053–1064

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Explicit formulas for partition pairs and triples with 3-cores

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ARTICLE INFO

Article history:
Received 16 July 2015
Available online 1 October 2015
Submitted by B.C. Berndt

Dedicated to my beloved mother, father, brother and sisters

Keywords:
Partitions
3-Cores
Ramanujan's ${}_1\psi_1$ summation
Bailey's ${}_6\psi_6$ formula

ABSTRACT

Let $A_3(n)$ (resp. $B_3(n)$) denote the number of partition pairs (resp. triples) of n where each partition is 3-core. By applying Ramanujan's ${}_1\psi_1$ formula and Bailey's ${}_6\psi_6$ formula, we find the explicit formulas for $A_3(n)$ and $B_3(n)$. Using these formulas, we confirm a conjecture of Xia and establish many arithmetic identities satisfied by $A_3(n)$ and $B_3(n)$.

1. Introduction

A partition of a positive integer n is any nonincreasing sequence of positive integers whose sum is n. For example, $6 = 3 + 2 + 1$ and $\lambda = \{3, 2, 1\}$ is a partition of 6. A partition λ of n is said to be a t-core if it has no hook numbers that are multiples of t. We denote the number of t-core partitions of n by $a_t(n)$.

The generating function of $a_t(n)$ is given by (see [6, Eq. (2.1)])

$$\sum_{n=0}^{\infty} a_t(n)q^n = \frac{(q^t;q^t)_\infty^t}{(q;q)_\infty}, \tag{1.1}$$

here and throughout this paper, we use the following notation

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n), \quad (a;q)_n := \frac{(a;q)_\infty}{(aq^n;q)_\infty} \quad (-\infty < n < \infty).$$

For convenience, we also introduce the brief notation

$$(a_1, a_2, \cdots, a_n; q)_\infty := (a_1;q)_\infty (a_2;q)_\infty \cdots (a_n;q)_\infty.$$

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http://dx.doi.org/10.1016/j.jmaa.2015.09.074

A partition k-tuple $(\lambda_1, \lambda_2, \cdots, \lambda_k)$ of n is a k-tuple of partitions $\lambda_1, \lambda_2, \cdots, \lambda_k$ such that the sum of all the parts equals n. For example, let $\lambda_1 = \{2, 1\}$, $\lambda_2 = \{1, 1\}$, $\lambda_3 = \{1\}$. Then (λ_1, λ_2) is a partition pair of 5 since $2 + 1 + 1 + 1 = 5$, and $(\lambda_1, \lambda_2, \lambda_3)$ is a partition triple of 6 since $2 + 1 + 1 + 1 + 1 = 6$. A partition k-tuple of n with t-cores is a partition k-tuple $(\lambda_1, \lambda_2, \cdots, \lambda_k)$ of n where each λ_i is t-core for $i = 1, 2, \cdots, k$.

Let $A_t(n)$ (resp. $B_t(n)$) denote the number of partition pairs (resp. triples) of n with t-cores. From (1.1) we know the generating functions for $A_t(n)$ and $B_t(n)$ are

$$\sum_{n=0}^{\infty} A_t(n) q^n = \frac{(q^t; q^t)_\infty^{2t}}{(q; q)_\infty^2} \tag{1.2}$$

and

$$\sum_{n=0}^{\infty} B_t(n) q^n = \frac{(q^t; q^t)_\infty^{3t}}{(q; q)_\infty^3} \tag{1.3}$$

respectively.

In this paper, we focus on partition k-tuples with 3-cores for $1 \leq k \leq 3$. The properties of $a_3(n)$, $A_3(n)$ and $B_3(n)$ have drawn much attention in the past years. In 1996, using the tools of modular forms, Granville and Ono [8] first discovered the following formula for $a_3(n)$:

$$a_3(n) = d_{1,3}(3n + 1) - d_{2,3}(3n + 1), \tag{1.4}$$

where $d_{r,3}(n)$ denote the number of positive divisors of n congruent to r modulo 3.

In 2009, by using some known identities, Hirschhorn and Sellers [9] provided an elementary proof of (1.4). Moreover, let

$$3n + 1 = \prod_{p_i \equiv 1 \pmod 3} p_i^{\alpha_i} \cdot \prod_{q_j \equiv 2 \pmod 3} q_j^{\beta_j}$$

with each $\alpha_i, \beta_j \geq 0$ be the prime factorization of $3n + 1$, they gave the explicit formula:

$$a_3(n) = \begin{cases} \prod(\alpha_i + 1) & \text{if all } \beta_j \text{ are even;} \\ 0 & \text{otherwise.} \end{cases}$$

Some arithmetic identities were then obtained as corollaries. For example, let $p \equiv 2 \pmod 3$ be a prime and let k be a positive even integer. Then, for all $n \geq 0$,

$$a_3\left(p^k n + \frac{p^k - 1}{3}\right) = a_3(n).$$

In 2014, Lin [10] found some arithmetic relations about $A_3(n)$ such as $A_3(8n + 6) = 7A_3(2n + 1)$. By using some theta function identities, Baruah and Nath [4] established three infinite families of arithmetic identities involving $A_3(n)$. For any integer $k \geq 1$, they proved that

$$\begin{aligned}
A_3\left(2^{2k+2} n + \frac{2(2^{2k} - 1)}{3}\right) &= \frac{2^{2k+2} - 1}{3} A_3(4n), \\
A_3\left(2^{2k+2} n + \frac{2(2^{2k+2} - 1)}{3}\right) &= \frac{2^{2k+2} - 1}{3} A_3(4n + 2) - \frac{2^{2k+2} - 4}{3} A_3(n), \\
A_3\left(2^{2k+1} n + \frac{5 \cdot 2^{2k} - 2}{3}\right) &= \left(2^{2k+1} - 1\right) A_3(2n + 1).
\end{aligned} \tag{1.5}$$