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# Spectra and variance of quantum random variables

Douglas Farenick [a,*], Michael J. Kozdron [a], Sarah Plosker [b]

[a] Department of Mathematics and Statistics, University of Regina, Regina, Saskatchewan S4S 0A2, Canada

[b] Department of Mathematics and Computer Science, Brandon University, Brandon, Manitoba R7A 6A9, Canada

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We study essentially bounded quantum random variables and show that the Gelfand spectrum of such a quantum random variable $\psi$ coincides with the hypoconvex hull of the essential range of $\psi$. Moreover, a notion of operator-valued variance is introduced, leading to a formulation of the moment problem in the context of quantum probability spaces in terms of operator-theoretic properties involving semi-invariant subspaces and spectral theory. As an application of quantum variance, new measures of random and inherent quantum noise are introduced for measurements of quantum systems, modifying some recent ideas of Polterovich [17].



## 1. Introduction

Some of the most basic and useful properties of classical random variables are altered when passing from real- or complex-valued measurable functions to operator-valued measurable functions (that is, from classical to quantum random variables). In earlier works [8,9,12], a certain operator-valued formulation of the notion of expectation of a quantum random variable was considered. In the present paper, we consider a similar formulation for the variance of a quantum random variable. As in these earlier investigations, the noncommutativity of operator algebra will lead to some structure that simply does not appear in the classical setting.

It is a basic fact of functional analysis that the essential range of an essentially bounded random variable coincides with the spectrum of a certain element in an abelian von Neumann algebra. Specifically, if $\psi : X \to \mathbb{C}$ is an essentially bounded function on a probability space $(X, \mathcal{F}(X), \mu)$, then the essential range of $\psi$ is precisely the spectrum of $\psi$, where one considers $\psi$ as an element of the von Neumann algebra

* Corresponding author.
*E-mail addresses:* douglas.farenick@uregina.ca (D. Farenick), kozdron@stat.math.uregina.ca (M.J. Kozdron), ploskers@brandonu.ca (S. Plosker).

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$L^\infty(X, \mathcal{F}(X), \mu)$. We will arrive at a similar result for essentially bounded quantum random variables on quantum probability spaces using higher dimensional spectra. However, it will turn out that our investigation of quantum variance will also involve notions from spectral theory. In particular, the quantum moment problem admits a characterisation entirely within spectral terms.

As an application of our operator-valued variance, we consider some recent work of Polterovich [17] on random and inherent quantum noise in which the variance has a role. In Polterovich's work, a somewhat hybrid context is at play: while the measures are operator-valued, the random variables are classical. In modifying Polterovich's ideas to account for operator-valued measures and operator-valued random variables, we formulate new measures of quantum noise. One of the main consequences of our results in this direction is that if an experimental apparatus is free of random quantum noise, then it is classical, not quantum mechanical. Our work on quantum noise involves another idea that may be of value in other settings, namely that of quantum randomisation (or smearing), which is in contrast to the hybrid notion of smearing studied in early works such as [4,11]. By way of quantum randomisation, we also modify another concept of Polterovich to obtain a measure of the intrinsic quantum noise of the apparatus represented by $\nu$.

If $(X, \mathcal{F}(X))$ denotes an arbitrary measurable space, and if $M$ is a von Neumann algebra with predual $M_*$ and positive cone $M_+$, then a function $\nu : \mathcal{F}(X) \to M$ is a positive operator-valued measure (POVM) if

1. $\nu(E) \in M_+$ for every $E \in \mathcal{F}(X)$,
2. $\nu(X) \neq 0$, and
3. $\omega \circ \nu : \mathcal{F}(X) \to \mathbb{C}$ is a complex measure for every $\omega \in M_*$.

Note that the third condition above asserts that, for every countable collection $\{E_k\}_{k\in\mathbb{N}} \subseteq \mathcal{F}(X)$ with $E_j \cap E_k = \emptyset$ for $j \neq k$,

$$\nu\left(\bigcup_{k\in\mathbb{N}} E_k\right) = \sum_{k\in\mathbb{N}} \nu(E_k), \tag{1}$$

where the convergence is with respect to the ultraweak topology of $M$.

If a POVM $\nu$ also satisfies $\nu(E \cap F) = \nu(E)\nu(F)$ for all $E, F \in \mathcal{F}(X)$, then $\nu$ is called a projective POVM. An important theorem of M.A. Neumark [14], [15, Theorem 4.6] states that every POVM admits a dilation to a projective POVM. Lastly, if a POVM $\nu$ has the property that $\nu(X) = 1$, the identity element of $M$, then $\nu$ is called a quantum probability measure.

A function $\psi : X \to M$ is said to be measurable if the complex-valued function $\omega\circ f$ on $X$ is measurable for every $\omega \in M_*$. Furthermore, if $\nu$ is a quantum probability measure, then a measurable function $\psi : X \to M$ is called a quantum random variable.

Suppose that $\omega \in M_*$ is a faithful state on $M$ and that $\nu$ is a quantum probability measure. Then $\omega \circ \nu$ is a (classical) probability measure and, because $\omega$ is faithful, $\nu$ and $\omega \circ \nu$ are mutually absolutely continuous. The predual of the von Neumann algebra $L^\infty(X, \omega \circ \nu)\overline{\otimes} M$ is given by $L^1_{M_*}(X, \omega \circ \nu)$ [20, Theorem IV.7.17]. By way of this duality isomorphism, if $\Psi \in L^\infty(X, \omega \circ \nu)\overline{\otimes} M$, then there is a bounded measurable function $\psi : X \to M$ such that, for each $f \in L^1_{M_*}(X, \omega \circ \nu)$, the complex number $\Psi(f)$ is given by

$$\Psi(f) = \int_X \omega\left(f(x)\psi(x)\right) \, \mathrm{d}(\omega \circ \nu)(x).$$

Although $\psi$ is not unique, it is unique up to a set of $\omega \circ \nu$-measure zero. We therefore identify $\Psi$ and $\psi$ and consider the elements of $L^\infty(X, \omega \circ \nu)\overline{\otimes} M$, in the case where $\nu$ is a quantum probability measure, to be bounded quantum random variables $\psi : X \to M$.

The general context described above for operator-valued measures and functions is considered in this paper only in the setting a finite factor $M$ of type $\mathrm{I}_d$; that is, $M = \mathcal{B}(\mathcal{H})$ for some $d$-dimensional Hilbert