J. Math. Anal. Appl. 434 (2016) 1211–1239

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Global analysis of an infection age model with a class of nonlinear incidence rates

Bentout Soufiane, Tarik Mohammed Touaoula *

Laboratoire d'Analyse Non-Linéaire et Mathématiques Appliquées and Department of Mathematics, University Aboubekr Belkaïd, Tlemcen 13000, Algeria

A R T I C L E   I N F O

Article history:
Received 2 December 2014
Available online 3 October 2015
Submitted by J. Shi

Keywords:
Epidemiology
Nonlinear incidence
Compact attractor
Uniform persistence
Lyapunov functional
Global stability

A B S T R A C T

SIR infection age models with a very general class of nonlinear incidence rates $f(S,J)$ are investigated. We give a necessary and sufficient condition for global asymptotic stability of the free-equilibrium related to the basic reproduction number. Furthermore, additional conditions allow us to prove an exponential stability of this disease-free equilibrium. Finally, by using a Lyapunov functional, we show the global asymptotic stability of the endemic equilibrium whenever it exists.

1. Introduction

Dynamic structured population models make distinction between individuals according to some particular aspects like birth, date, etc. One of the most important of them is the age of infection; that is how long the infection began.

Another important aspect to be considered is the incidence rate, that is, newly infected individuals per unit of time. The convenient choice of the functional form of the incidence rate ensures that the model gives a good qualitative description for the disease dynamics under consideration.

In the literature of mathematical modeling, very frequently a bilinear form of incidence rate is used such as, SI or SI/N, where S represents susceptible individuals and I the infected individuals (N is the total size of population); this is done to characterize the fact that the contact number between susceptible and infective is proportional to the product of both sub populations.

When considering a system of ordinary differential equations, Capasso et al. [5], used for incidence rate the nonlinear functional $\beta \frac{S(t)I(t)}{1+\alpha I(t)}$, to model the spread of Cholera in Bari in 1973.

* Corresponding author.
E-mail addresses: bentoutsofiane@gmail.com (B. Soufiane), tarik.touaoula@mail.univ-tlemcen.dz (T.M. Touaoula).

http://dx.doi.org/10.1016/j.jmaa.2015.09.066

The first most general case where incidence rate was described in terms of a general function $f(S, I)$ was introduced by Feng and Thieme [9,10]. Thereafter, Korobeinikov et al. [17] formulated a variety of models with incidence rates having the form $f(S(t))G(I(t))$ and Korobeinikov [15,16] obtained some global properties of basic *SIR* and *SIRS* epidemic models with more general framework of the incidence rate $f(S(t), I(t))$.

To study the impact of the time delays, (used for instance, to model the latency of the infection in a host) many authors proposed and analyzed *SIR*, *SEIR* models with discrete delay and bilinear incidence, see [19,24], and the references therein. Delay epidemic models with nonlinear incidence were studied in [34,36], however, only global stability of the disease free equilibrium was proved. Huang et al. made a breakthrough, firstly by studying in [14] the nonlinear model

$$\begin{cases} S'(t) = \mu - F(S(t))G(i(t-\tau)) - \mu S(t), \\ i'(t) = F(S(t))G(i(t-\tau)) - (\mu+\sigma)i(t), \\ r'(t) = \sigma i(t) - \mu r(t), \end{cases} \tag{1.1}$$

where the incidence took the form $F(S)G(I)$, and assumed that F is increasing and G is concave. Later in [13], Huang et al. considered the more general nonlinear incidence rate of the form $f(S(t), I(t-\tau))$. In these both papers, global stability of endemic and disease free equilibria was established by constructing suitable Lyapunov functionals.

Beretta et al. [1], introduced and studied a SIR model with time distributed delays, for a vector-born diseases. Thereafter, various epidemiological models, have been proposed and studied, [3,12,20,23–25,29] and the references therein.

In the recent paper [24], McCluskey investigated the following model

$$\begin{cases} S'(t) = b - \beta S(t)\int_0^h f(\tau)i(t-\tau)d\tau - \mu_1 S(t) \\ i'(t) = \beta S(t)\int_0^h f(\tau)i(t-\tau)d\tau - (\mu_2+\lambda)i(t) \\ r'(t) = \lambda i(t) - \mu_3 r(t). \end{cases}$$

By using a Lyapunov functional, the global dynamics are fully determined for $R_0 > 1$. More precisely, it is proved in [24], that the endemic equilibrium is globally asymptotically stable whenever it exists.

In the context of age infection model, Thieme et al. [33] proposed and analyzed a model of the infection age-dependent infectivity. This work has been followed by many age structured models [2–4,6,21,32,35].

Recently, Magal et al. [21] have proposed and studied the following model

$$\begin{cases} S'(t) = \gamma - \nu_s S(t) - \eta S(t)\int_0^\infty \beta(a)i(t,a)da, \\ i_t(t,a) + i_a(t,a) = -\nu_I(a)i(t,a), \\ i(t,0) = \eta S(t)\int_0^\infty \beta(a)i(t,a)da, \\ i(0,a) = i_0(a), \quad S(0) = S_0, \end{cases}$$

where convenient Lyapunov functional was employed to obtain the stability of the endemic equilibrium.

Motivated by all these previous models, and by the huge existent epidemiological background in the literature, we propose in this paper a general model that incorporates both the infection age and a nonlinear incidence rate:

$$\begin{cases} S'(t) = A - \mu S(t) - f\big(S(t), J(t)\big), \quad t \geq 0, \\ i_t(t,a) + i_a(t,a) = -(\mu+\gamma(a))i(t,a), \\ J(t) = \int_0^\infty \beta(a)i(t,a)da, \end{cases} \tag{1.2}$$