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Exponential rate of periodic points and metric entropy in nonuniformly hyperbolic systems

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A R T I C L E   I N F O

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A B S T R A C T

For an ergodic hyperbolic measure ω of a C^{1+r} diffeomorphism f there is an ω-full measured set $\widetilde{\Lambda}$ such that for every invariant measure $\mu \in \mathcal{M}_{inv}(\widetilde{\Lambda}, f)$ the exponential growth rate of the number of periodic points whose atomic measures approximate μ equals to the metric entropy $h_\mu(f)$.

1. Introduction

It is known that the shadowing property of differential dynamical systems with some hyperbolicity yields many periodic points in the state space. Besides, there are so many periodic points that fruitful properties of hyperbolic measures can be well approximated by periodic measures, for instance, the Lyapunov exponents [16] and Oseledets splittings [9]. As a classical result in uniformly hyperbolic systems, Sigmund [14] proved that periodic measures are dense in the set of invariant measures. For the non-uniformly hyperbolic case, periodic measures are dense in the set of invariant measures supported on a total measurable set with respect to a hyperbolic ergodic measures [4,9].

Katok [6] stated that, if f is a C^{1+r} diffeomorphism of a compact surface with positive topological entropy, then the exponential growth rate of periodic orbits is bounded from below by entropy. In 2010, Liao, Sun and Tian [10] proved that, the metric entropy of a **hyperbolic ergodic** measure can be approximated by the exponential growth rate of periodic measures, which establishes Katok's inequality to be equality. Our goal in the current paper is to present the equality for the metric entropy of **invariant** measures, which is, sometimes, **not ergodic**, in terms of the exponential growth rate of periodic orbits and thus generalize the results in [10]. The periodic points employed in the present paper are different from those in the ergodic

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case as in [10], since we require them to approximate the underlying invariant **non-ergodic** measures. The approach in the present paper is thus different from that in [10].

Now we start to introduce our results precisely. Let M be a compact connected boundary-less Riemannian manifold and $f : M \to M$ a C^{1+r} diffeomorphism. An ergodic measure is called hyperbolic if it is a measure without zero Lyapunov exponents. For positive numbers $\alpha, \beta \gg \varepsilon$, and integer $k > 0$, the hyperbolic block $\Lambda_k = \Lambda_k(\alpha, \beta; \varepsilon)$ with index k is understood to be the collection of such points $x \in M$ that there is a Df-invariant decomposition on the tangent bundle $T_x M = E^s_x \oplus E^u_x$ satisfying:

- $\|Df^n|E^s_{f^m x}\| \leq e^{\varepsilon k} e^{-(\beta-\varepsilon)n} e^{\varepsilon|m|}$, $\forall\, m \in \mathbb{Z}$, $n \geq 1$;
- $\|Df^{-n}|E^u_{f^m x}\| \leq e^{\varepsilon k} e^{-(\alpha-\varepsilon)n} e^{\varepsilon|m|}$, $\forall\, m \in \mathbb{Z}$, $n \geq 1$; and
- $\tan(\angle(E^s_{f^m x}, E^u_{f^m x})) \geq e^{-\varepsilon k} e^{-\varepsilon|m|}$, $\forall\, m \in \mathbb{Z}$.

From the definition we can see the index k indicates the level of hyperbolicity on Λ_k, and the larger k is, the weaker the hyperbolicity is.

Denote the union of all Λ_k, calling a nonuniformly hyperbolic set or Pesin set, by

$$\Lambda(\alpha, \beta; \varepsilon) = \bigcup_{k=1}^{+\infty} \Lambda_k(\alpha, \beta; \varepsilon).$$

It is easy to verify that $\omega(\Lambda(\alpha, \beta; \varepsilon)) = 1$ provided the ergodic measure ω has no Lyapunov exponents in $[-\beta, \alpha]$. We fix ω to be such an ergodic hyperbolic measure and denote by $\omega\,|_{\Lambda_k}$ the conditional measure of ω on Λ_k. Set $\widetilde{\Lambda}_k = \mathrm{supp}(\omega\,|_{\Lambda_k})$ and $\widetilde{\Lambda} = \cup_{k\geq 1}\widetilde{\Lambda}_k$. Clearly, $f^{\pm}(\widetilde{\Lambda}_k) \subset \widetilde{\Lambda}_{k+1}$. Moreover, $\widetilde{\Lambda}$ is f-invariant with ω-full measure. In fact many systems enjoy the structure of $\widetilde{\Lambda}$ with many non-trivial invariant measures, for example the non-uniformly hyperbolic diffeomorphisms described by Katok [5].

Given an f-invariant set Γ, denote by $\mathcal{M}_{inv}(\Gamma, f)$ the set of all f-invariant measures with full measure on Γ. Assume $\{\varphi_i\}_{i=1}^{\infty}$ is a dense subset of $C^0(M)$ giving the weak* topology, that is,

$$D(\mu,\, m) = \sum_{i=1}^{\infty} \frac{|\int \varphi_i d\mu - \int \varphi_i dm|}{2^{i+1}\|\varphi_i\|}$$

for $\mu, m \in \mathcal{M}_{inv}(M, f)$. It is easy to verify the affine property of D, i.e., for any $\mu,\, m_1,\, m_2 \in \mathcal{M}_{inv}(M, f)$ and $0 \leq \theta \leq 1$,

$$D(\mu,\, \theta m_1 + (1-\theta)m_2) \leq \theta D(\mu, m_1) + (1-\theta)D(\mu, m_2).$$

In addition, $D(\mu, m) \leq 1$ for any $\mu, m \in \mathcal{M}_{inv}(M, f)$. Given $x \in M$, define the n-ordered empirical measure of x by

$$\mathcal{E}_n(x) = \frac{1}{n}\sum_{i=0}^{n-1} \delta_{f^i(x)},$$

where δ_y is the Dirac mass at $y \in M$. Given an invariant measure μ, a point x is called a generic point with respect to μ if

$$\lim_{n\to+\infty} \mathcal{E}_n(x) = \mu.$$

By Birkhoff Theorem the set of generic points x with $\mathcal{E}_n(x)$ converging to an **ergodic** measure has full measure but by Ergodic Decomposition Theorem the set of generic points x with $\mathcal{E}_n(x)$ converging to an invariant **non-ergodic** measure has zero measure.