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Approximate helices of continuous iteration semigroups

J. Chudziak [a,*], G. Guzik [b]

[a] Faculty of Mathematics and Natural Sciences, University of Rzeszów, Prof. S. Pigonia 1, 35-959 Rzeszów, Poland
[b] Faculty of Applied Mathematics, AGH University of Science and Technology, Mickiewicza 30, 30-059 Kraków, Poland

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ABSTRACT

We show that every approximate helix of a continuous iteration semigroup on a closed real interval can be approximated by some helix of the same iteration semigroup.

1. Introduction

Consider a function $H : (0, \infty) \times X \to Y$ satisfying the following functional equation

$$H(s + t, x) = H(t, F(s, x)) + H(s, x) \quad \text{for } s, t \in (0, \infty),\ x \in X \tag{1}$$

where, in general, $F : (0, \infty) \times X \to X$ is a given function, X is a nonempty set and $(Y, +)$ is an Abelian group. Equation (1) is mostly studied in the case when F satisfies the translation equation, i.e.

$$F(s + t, x) = F(t, F(s, x)) \quad \text{for } s, t \in (0, \infty),\ x \in X. \tag{2}$$

In fact, it can be shown that this is the only reasonable case. Every solution of equation (1) is called a *helix* of F (or over F).

The notion of helices is strictly connected with properties of some curves and it was introduced by Kolmogorov [12] in the framework of stochastic integrals and stochastic differential equations (see also [9]). It also appears in the theory of stationary processes (see [2, Appendix A3] and related references therein),

* Corresponding author.
E-mail addresses: chudziak@ur.rzeszow.pl (J. Chudziak), guzik@agh.edu.pl (G. Guzik).

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it is used for changing velocity in flows (see [15]) and in the theory of linear operators (see [8]). Equation (1) is a special kind of a more general cocycle equation, namely when the Abelian group $(Y, +)$ is replaced by a semigroup. In particular, when it is a suitable semigroup of functions with the usual composition operation, the use of such cocycle mappings is fundamental in the theory of random as well as nonautonomous dynamical systems. The reader interested in this subject can find a comprehensive lecture on the theory of such systems and a vast literature in [2] and [11].

The solutions of the cocycle equation in the case where F is a continuous iteration semigroup on a closed real interval and a cocycle mapping has its values in an arbitrary Abelian group have been determined in [7]. The analogous results for an arbitrary group have been obtained recently in [5], where the cocycles of continuous iteration semigroups have been applied in construction of continuous skew-product semiflows defined on a closed real rectangle.

In this paper we consider the approximate helices. More precisely, we deal with the solutions $H : (0, \infty) \times X \to Y$ of the inequality

$$\|H(s + t, x) - H(t, F(s, x)) - H(s, x)\| \leq \varepsilon \text{ for } s, t \in (0, \infty),\ x \in X \tag{3}$$

where X is an arbitrary closed interval contained in $[-\infty, \infty]$, $F : (0, \infty) \times X \to X$ is a given continuous iteration semigroup, i.e. a solution of the translation equation (2) continuous with respect to both variable, $(Y, \| \cdot \|)$ is a Banach space and ε is a fixed nonnegative real number. Our main results state that every approximate helix of a continuous iteration semigroup can be approximated by an exact one. We also show that the analogous result holds for helices of some special skew-product flows strictly connected with continuous iteration groups on an interval.

It is remarkable that approximate iterations groups and semigroups have been already studied in [3], [13] and [14].

2. Continuous iteration semigroups and groups

The full description of continuous iteration semigroups in the case where X is a closed real interval belongs to M.C. Zdun (see [16]). Let us briefly recall some facts from [16] which will be useful in our considerations. Let $X \subset [-\infty, \infty]$ be a closed interval and let $F : (0, \infty) \times X \to X$ be a continuous iteration semigroup. Put

$$f := F(1, \cdot).$$

Then there are points $x_1, x_2 \in X$ such that

$$x_1 \leq f(x_1) \leq f(x) \leq f(x_2) \leq x_2 \quad \text{for } x \in X,$$

the restriction $f|_{[x_1,x_2]}$ is increasing and any interval of constancy of f contains a fixed point of f. Since the set $\text{Fix}\, f := \{x \in X : f(x) = x\}$ is closed in X there is a countable family $\{X_j : j \in J\}$ of pairwise disjoint and open in $[x_1, x_2]$ intervals such that

$$\bigcup_{j \in J} X_j = [x_1, x_2] \backslash \text{Fix}\, f.$$

Moreover, a function $e : X \to X$ given by

$$e(x) = \lim_{t \to 0^+} F(t, x)$$

satisfies the following conditions: