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On certain geometric properties of polyharmonic mappings ☆

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ABSTRACT

The main purpose of this paper is twofold. First, the three-quarter theorem is established for a large class of polyharmonic mappings. Second, a class of polyharmonic mappings with positive Jacobian and being univalent in the unit disk $\mathbb{D}$ are constructed.

1. Introduction

A $2p$-continuously differentiable complex-valued function $F = u + iv$ in a domain $D \subset \mathbb{C}$ is *polyharmonic (or p-harmonic)* if F satisfies the p-harmonic equation

$$\Delta^p F := \Delta\big(\Delta^{p-1}\big) F = 0.$$

When $p = 1$ (resp. $p = 2$), F is harmonic (resp. biharmonic). In particular, a harmonic mapping in a simply connected domain D of $\mathbb{C}$ can be expressed in the form $f = g + \overline{h}$, where h and g are analytic in D. See [6,8,9] for more properties of harmonic mappings.

In the following, we always assume that the domain $D \subset \mathbb{C}$ is the unit disk $\mathbb{D} = \{z \in \mathbb{C} : \ |z| < 1\}$. That is, throughout this paper, we consider polyharmonic mappings on $\mathbb{D}$.

The subsequent proposition provides a characterization of polyharmonic mappings which is essential to our analysis (cf. [5]).

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Proposition 1.1. *If F is a polyharmonic mapping in D then it has the representation:*

$$F = F(z, \bar{z}) = \sum_{n=0}^{p-1} r^{2n} F_n,$$

where F_n's are complex-valued harmonic mappings in D and $r^{2n} = |z|^{2n} = (z\bar{z})^n$.

In particular, if F is a biharmonic mapping then it has the representation

$$F = r^2 G + H = |z|^2 G + H,$$

where G and H are complex-valued harmonic mappings in $\mathbb{D}$.

A set $E \subset \mathbb{C}$ is *starlike* with respect to a point $w_0 \in E$ if and only if the linear segment joining w_0 to every other point $w \in E$ lies entirely in E, while a set E is said to be *convex* if and only if it is starlike with respect to each of its points, that is, if and only if the linear segment joining any two points of E lies entirely in E.

A univalent polyharmonic mapping F, with $F(0) = 0$, is said to be *starlike* in $\mathbb{D}$ if the domain bounded by the curve $F(re^{it})$ is starlike with respect to the origin for each $0 < r < 1$. In other words, F is starlike in $\mathbb{D}$ if

$$\frac{\partial \arg F(re^{it})}{\partial t} = \mathrm{Re}\left(\frac{zF_z - \bar{z}F_{\bar{z}}}{F}\right) > 0$$

for all $z \neq 0$ and $r \in (0, 1)$.

A univalent polyharmonic mapping F, with $F(0) = 0$ and $\frac{\partial F(re^{it})}{\partial t} \neq 0$ for $0 < r < 1$, is said to be *convex* if the domain bounded by the curve $F(re^{it})$ is convex for each $0 < r < 1$. Putting it differently, F is convex if

$$\frac{\partial \arg \frac{\partial}{\partial t} F(re^{it})}{\partial t} > 0$$

for all $z \neq 0$ and $r \in (0, 1)$.

It is known that a harmonic mapping F is locally univalent if the Jacobian of F, denoted by J_F,

$$J_F = |F_z|^2 - |F_{\bar{z}}|^2 \neq 0,$$

and F is orientation preserving if

$$J_F = |F_z|^2 - |F_{\bar{z}}|^2 > 0.$$

Let S, see [8], denote the class of all conformal univalent functions on $\mathbb{D}$ of the form

$$G(z) = z + \sum_{k=2}^{\infty} a_k z^k.$$

In this paper, we mainly consider the following class of polyharmonic mappings together with its two subclasses, where $p \geq 2$, which are as follows:

$$\mathcal{PH} = \left\{ F : \ F = \sum_{n=0}^{p-1} \lambda_n r^{2n}\, G, \ \text{where } G \text{ is harmonic, all } \lambda_i\text{'s are real constants,} \right.$$
$$\left. \lambda_1^2 + \lambda_2^2 + \ldots + \lambda_{p-1}^2 \neq 0 \text{ and } r = |z| \right\};$$