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# The subnormal completion problem in several variables

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ABSTRACT

The subnormal completion problem for $d$-variable weighted shifts is considered. Necessary and sufficient conditions are obtained for a collection of initial weights $\mathcal{C} = \{(\alpha_\gamma^{(1)}, \ldots, \alpha_\gamma^{(d)})\}_{\gamma \in \Gamma}$, where $\Gamma$ is from a family of finite indexing sets which includes $\{\gamma \in \mathbb{N}_0^d : 0 \leq |\gamma| \leq m\}$, to give rise to a $d$-variable subnormal weighted shift operator whose initial weights are given by $\mathcal{C}$. The conditions are communicated in terms of a new solution of a corresponding truncated $K$-moment problem. The case when $d = 2$ and all cubic moments are known is investigated in detail. In particular, using the solution of the subnormal completion problem in $d$-variables presented here, an easily checked concrete sufficient condition is given for a solution to the cubic subnormal completion problem in 2-variables and also an example of a collection of weights in 2-variables with cubic moments is provided which satisfies natural positivity conditions yet does not admit a subnormal completion.



## 1. Introduction

We present a new approach to the subnormal completion problem for weighted shift operators in $d$-variables. The approach is based on a solution to the truncated $K$-moment problem on $\mathbb{R}^d$ which originated in [18]. Given a Hilbert space $\mathcal{H}$, with inner product $\langle \cdot, \cdot \rangle$, we denote the space of bounded linear operators on $\mathcal{H}$ by $\mathcal{L}(\mathcal{H})$. An operator $A \in \mathcal{L}(\mathcal{H})$ is called normal and hyponormal if $A^*A - AA^* = 0$ and $A^*A - AA^* \succeq 0$, respectively. We call $A \in \mathcal{L}(\mathcal{H})$ subnormal if there exists a normal operator $B \in \mathcal{L}(\mathcal{K})$ so that $\mathcal{H} \subseteq \mathcal{K}$ and $A\xi = B\xi$ for $\xi \in \mathcal{H}$. The Bram–Halmos criterion [3] states that $A \in \mathcal{L}(\mathcal{H})$ is subnormal if and only if

$$\sum_{j,k=0}^{n} \langle A^j \xi_j, A^k \xi_k \rangle \geq 0 \tag{1.1}$$

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for all finite subsets $\{\xi_0, \dots, \xi_n\} \subset \mathcal{H}$. If $A, B \in \mathcal{L}(\mathcal{H})$, then $[A, B] \stackrel{\text{def}}{=} AB - BA$. We call an $\mathcal{L}(\mathcal{H})$-valued $d$-tuple of operators $T = (T_1, \dots, T_d)$ *jointly hyponormal* if

$$\begin{pmatrix} [T_1^*, T_1] & \cdots & [T_1^*, T_d] \\ \vdots & \ddots & \vdots \\ [T_d^*, T_1] & \cdots & [T_d^*, T_d] \end{pmatrix} \succeq 0. \tag{1.2}$$

We call an $\mathcal{L}(\mathcal{H})$-valued $d$-tuple $T = (T_1, \dots, T_d)$ *normal* if $[T_j, T_k] = 0$ and $T_j$ is normal for all $j, k = 1, \dots, d$. We call an $\mathcal{L}(\mathcal{H})$-valued $T = (T_1, \dots, T_d)$ *subnormal* if there exists a Hilbert space $\mathcal{K}$ and an $\mathcal{L}(\mathcal{K})$-valued normal $d$-tuple $U = (U_1, \dots, U_d)$, where $\mathcal{H} \subseteq \mathcal{K}$, so that $U_j\xi = T_j\xi$ for all $j = 1, \dots, d$ and $\xi \in \mathcal{H}$. It is readily seen that the following implications hold:

$$T \text{ is normal} \Rightarrow T \text{ is subnormal} \Rightarrow T \text{ is hyponormal.}$$

For more information on subnormal operators see [3] and [4]. See [1] and [5] for more information on joint hyponormality. For a multivariable generalization of the Bram–Halmos criterion see [12].

Let $\mathbb{N}_0^d$ denote the set of all $d$-tuples of nonnegative integers and $0_d = (0, \dots, 0) \in \mathbb{N}_0^d$. If $m = (m_1, \dots, m_d) \in \mathbb{N}_0^d$, then $|m| = \sum_{j=1}^d m_j$. A finite set $\Gamma \subset \mathbb{N}_0^d$ will be called a *lattice set* if every element of $\Gamma$ is path connected to $0_d$. For instance, $\{(0,0), (0,1), (1,1)\}$ is a lattice set, while $\{(0,0), (1,1)\}$ is not. Let $\ell_\infty(\mathbb{N}_0^d)$ denote the space of all bounded multisequences $\{a_m\}_{m\in\mathbb{N}_0^d}$ and $\ell_2(\mathbb{N}_0^d)$ be the Hilbert space of all multisequences $\{a_m\}_{m\in\mathbb{N}_0^d}$ so that $\sum_{m\in\mathbb{N}_0^d} |a_m|^2 < \infty$ with the usual inner product. Let $\{\varepsilon_m\}_{m\in\mathbb{N}_0^d}$ denote the canonical orthonormal basis of $\ell_2(\mathbb{N}_0^d)$ and $\mathrm{e}_j = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{N}_0^d$, where 1 is in the $j$th position.

If $\alpha^{(j)} = \{\alpha_m^{(j)}\}_{m\in\mathbb{N}_0^d} \in \ell_\infty(\mathbb{N}_0^d)$ for $j = 1, \dots, d$, then we define the $d$-variable weighted shift $T = (T_1, \dots, T_d)$ by

$$T_j\varepsilon_m \stackrel{\text{def}}{=} \alpha_m^{(j)}\varepsilon_{m+\mathrm{e}_j} \quad \text{for} \quad j = 1, \dots, d. \tag{1.3}$$

It is easy to check that $T_jT_k = T_kT_j$ if and only if

$$\alpha_{m+\mathrm{e}_j}^{(k)}\alpha_m^{(j)} = \alpha_{m+\mathrm{e}_k}^{(j)}\alpha_m^{(k)} \quad \text{for} \quad j, k = 1, \dots, d. \tag{1.4}$$

Given $m \in \mathbb{N}_0^d$, the moment of $\mathcal{C}_\infty = (\alpha^{(1)}, \dots, \alpha^{(d)})$ of order $m$ is given by

$$s_m = \begin{cases} 1 & \text{if} \quad m = 0_d, \\ (\alpha_{m-\mathrm{e}_j}^{(j)})^2 s_{m-\mathrm{e}_j} & \text{if} \quad m = (m_1, \dots, m_d) \quad \text{where} \quad m_j \neq 0. \end{cases} \tag{1.5}$$

It follows from (1.4) that $s_m$ can be computed via any nondecreasing path from $0_d$ to $m$.

**Problem 1.1** *(Subnormal completion problem in d-variables).* Given a finite collection of positive real numbers

$$\mathcal{C} = \{(\alpha_\gamma^{(1)}, \dots, \alpha_\gamma^{(d)})\}_{\gamma\in\Gamma}$$

which satisfies (1.4), where $\Gamma$ is a lattice set (i.e., a finite subset of $d$-tuples of nonnegative integers which are path connected to the origin), the *subnormal completion problem in d-variables* entails finding necessary and sufficient conditions for the existence of a subnormal $d$-variable weighted shift whose initial weights are given by $\mathcal{C}$. More precisely, given $\mathcal{C}$ as above, we wish to establish the existence of a $d$-variable weighted shift $T = (T_1, \dots, T_d)$ so that