J. Math. Anal. Appl. 434 (2016) 1549–1565

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Almost square Banach spaces

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A R T I C L E I N F O

Article history:
Received 4 February 2015
Available online 30 September 2015
Submitted by Richard M. Aron

Keywords:
Diameter two
Octahedral
M-ideal
Renorming
Intersection property
Almost square

A B S T R A C T

We single out and study a natural class of Banach spaces – almost square Banach spaces. In an almost square space we can find, given a finite set $x_1, x_2, \dots, x_N$ in the unit sphere, a unit vector y such that $\|x_i - y\|$ is almost one. These spaces have duals that are octahedral and finite convex combinations of slices of the unit ball of an almost square space have diameter 2. We provide several examples and characterizations of almost square spaces. We prove that non-reflexive spaces which are M-ideals in their biduals are almost square. We show that every separable space containing a copy of c_0 can be renormed to be almost square. A local and a weak version of almost square spaces are also studied.

1. Introduction

Let X be a Banach space with unit ball B_X, unit sphere S_X, and dual space X^*.

Definition 1.1. We will say that a Banach space X is

(i) *locally almost square* (LASQ) if for every $x \in S_X$ there exists a sequence $(y_n) \subset B_X$ such that $\|x \pm y_n\| \to 1$ and $\|y_n\| \to 1$.
(ii) *weakly almost square* (WASQ) if for every $x \in S_X$ there exists a sequence $(y_n) \subset B_X$ such that $\|x \pm y_n\| \to 1$, $\|y_n\| \to 1$ and $y_n \to 0$ weakly.
(iii) *almost square* (ASQ) if for every finite subset $(x_i)_{i=1}^N \subset S_X$ there exists a sequence $(y_n) \subset B_X$ such that $\|x_i \pm y_n\| \to 1$ for every $i = 1, 2, \dots, N$ and $\|y_n\| \to 1$.

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[1] The research of J. Langemets was supported by Estonian Science Foundation Grant 8976, Estonian Targeted Financing Project SF0180039s08, and Estonian Institutional Research Project IUT20-57.

http://dx.doi.org/10.1016/j.jmaa.2015.09.060
0022-247X/

Obviously WASQ implies LASQ, but it is not completely obvious that ASQ implies WASQ. This will be shown in Theorem 2.8. In the language of Schäffer [27, p. 31] a Banach space X is LASQ if and only if no $x \in S_X$ is *uniformly non-square* (see also [21, Proposition 2.2]).

The above definition was inspired by the following characterizations of octahedral norms shown by Haller, Langemets, and Põldvere in [15] (see Proposition 2.1, Proposition 2.4, and Lemma 3.1).

Proposition 1.2. *A Banach space X is said to be*

(i) locally octahedral *if for every $x \in S_X$ and every $\varepsilon > 0$ there exists $y \in S_X$ such that $\|x \pm y\| \geq 2 - \varepsilon$.*
(ii) weakly octahedral *if for every finite subset $(x_i)_{i=1}^N \subset S_X$, every $x^* \in B_{X^*}$, and every $\varepsilon > 0$ there exists $y \in S_X$ such that $\|x_i + ty\| \geq (1 - \varepsilon)(|x^*(x_i)| + t)$ for all $i = 1, 2, \ldots, N$ and $t > 0$.*
(iii) octahedral *if for every finite subset $(x_i)_{i=1}^N \subset S_X$ and every $\varepsilon > 0$ there exists $y \in S_X$ such that $\|x_i \pm y\| \geq 2 - \varepsilon$ for all $i = 1, 2, \ldots, N$.*

Our main interest in the properties LASQ, WASQ, and ASQ comes from their relation to diameter two properties. Let X be a Banach space. Recall that a *slice* of B_X is a set of the form

$$S(x^*, \alpha) = \{x \in B_X : x^*(x) > 1 - \alpha\},$$

where $x^* \in S_{X^*}$ and $\alpha > 0$. According to [1] X has the *local diameter* 2 *property (LD2P)* if every slice of B_X has diameter 2, X has the *diameter* 2 *property (D2P)* if every nonempty relatively weakly open subset of B_X has diameter 2, and X has the *strong diameter* 2 *property (SD2P)* if every finite convex combination of slices of B_X has diameter 2. The following theorem is shown in [15] (see Theorems 2.3, 2.7, and 3.3).

Theorem 1.3. *Let X be a Banach space. Then*

(i) *X has the LD2P if and only if X^* is locally octahedral.*
(ii) *X has the D2P if and only if X^* is weakly octahedral.*
(iii) *X has the SD2P if and only if X^* is octahedral.*

The connection between the SD2P and octahedrality has also been studied in [7].

The starting point of this paper was the observation by Kubiak that if X is LASQ then X has the LD2P and similarly if X is WASQ then X has the D2P (see Propositions 2.5 and 2.6 in [21]). The basic idea from Kubiak's proof can be used with a result from [1] to show that ASQ spaces have the SD2P, but we will give a shorter self-contained proof that X^* is octahedral whenever X is ASQ in Proposition 2.5.

It is known that the three diameter 2 properties are different (see [6,3,14]). A natural question is whether LASQ, WASQ, and ASQ are different properties. A consequence of Example 3.3 is that $L_1[0,1]$ is a WASQ space which is not ASQ.

We now give a short outline of the paper. Section 2 starts with a few characterizations of LASQ and ASQ. In Lemma 2.6 we show that ASQ spaces have to contain almost isometric copies of c_0. This in turn is used to prove Theorem 2.8, which shows that ASQ implies WASQ. The final result in this section is Theorem 2.9, where we show that every Banach space containing a complemented subspace isomorphic to c_0 can be equivalently renormed to be ASQ.

In Section 3 we will give examples of spaces which are LASQ, WASQ, and ASQ.

In Section 4 we show that non-reflexive spaces which are M-ideals in their biduals are ASQ (see Theorem 4.2). However, the class of ASQ spaces is much bigger than the class of spaces that are M-ideals in their biduals (see the discussion following Corollary 4.3 and Example 6.4).

In Section 5 we study the stability of both (local/weak) octahedrality and (L/W)ASQ when forming absolute sums of Banach spaces. We show that local and weak octahedral, LASQ, and WASQ spaces have