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# Isometries of the Toeplitz matrix algebra ☆

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ABSTRACT

We study the structure of isometries defined on the algebra $\mathcal{A}$ of upper-triangular Toeplitz matrices. Our first result is that a continuous multiplicative norm preserving map $\mathcal{A} \to M_n(\mathbb{C})$ must be of the form either $A \mapsto UAU^*$ or $A \mapsto U\overline{A}U^*$, where $\overline{A}$ is the complex conjugation and $U$ is a unitary matrix. In our second result we use a range of ideas in operator theory and linear algebra to show that every linear isometry $\mathcal{A} \to M_n(\mathbb{C})$ is of the form $A \mapsto UAV$ where $U$ and $V$ are two unitary matrices. This implies, in particular, that every such an isometry is a complete isometry and that a unital linear isometry $\mathcal{A} \to M_n(\mathbb{C})$ is necessarily an algebra homomorphism.



## 1. Introduction

The $n \times n$ upper-triangular Toeplitz matrices over the field of complex numbers form a unital abelian subalgebra $\mathcal{A}$ of the algebra $M_n(\mathbb{C})$ of all $n \times n$ complex matrices. Our concern in this paper is with the structure of linear isometric maps $\varphi : \mathcal{A} \to M_n(\mathbb{C})$, where the norm of a matrix $X \in M_n(\mathbb{C})$ is the norm induced by considering $X$ as a linear operator on the Hilbert space $\mathbb{C}^n$ with respect to the standard inner product. That there might be something of interest to deduce about such linear maps is suggested by a result of Farenick, Gerasimova, and Shvai [12] which arose from their study of complete unitary-similarity invariants for certain complex matrices. Their result states that if $\varrho : \mathcal{A} \to M_n(\mathbb{C})$ is a unital isometric homomorphism, then there is a unitary $U \in M_n(\mathbb{C})$ such that $\varrho(X) = UXU^*$ for every $X \in \mathcal{A}$. In other words, every unital isometric homomorphism of the Toeplitz algebra $\mathcal{A}$ back into $M_n(\mathbb{C})$ extends to an isometric automorphism of the algebra $M_n(\mathbb{C})$. As a consequence of the results of the present paper, this

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conclusion is also true for unital isometric maps that are merely linear. Hence, if a unital linear map $\varphi : \mathcal{A} \to M_n(\mathbb{C})$ fails to be multiplicative, then the map cannot be an isometry. A similar conclusion is true for continuous maps that are multiplicative but not necessarily linear.

Every linear isometric map of an operator space into $M_n(\mathbb{C})$ is completely bounded [26, Proposition 8.11], but there are many examples of linear isometries that fail to be completely isometric—the transpose map on $M_n(\mathbb{C})$ being the most famous basic example. However, the restriction of the transpose map to $\mathcal{A}$ is completely isometric and it is a consequence of our work here that every linear isometric map $\varphi : \mathcal{A} \to M_n(\mathbb{C})$ is completely isometric. Thus, the results of this paper align with other results in which linear isometries of operator algebras are necessarily completely isometric (for example, the relevant results on isometries of nest algebras and reflexive algebras in [2,22,23,25]).

There is a vast literature on the structure of maps defined on the algebra of complex $n \times n$ matrices that preserve some properties of interest (such as the norm of a matrix, the spectrum, the rank, and so forth). A sample list of papers devoted to "preserver problems" is [5,8,9,16,20,18,19,21,29]. Many such results depend on the use of matrix units or the abundance of rank-1 matrices in the full matrix algebra $M_n(\mathbb{C})$. Our contribution in this direction is rather novel in that we consider linear maps on a small subspace of matrices with limited structure and which has just one (up to scalar multiple) rank-1 matrix and matrix unit.

Our main results in this paper are Theorem 2.5 and Theorem 5.1. In the first result we show that every continuous multiplicative isometry $\mathcal{A} \to M_n$ is of the form $A \mapsto UAU^*$ or of the form $A \mapsto U\overline{A}U^*$, where $U$ is a unitary matrix and $\overline{A}$ denotes the complex conjugation. In our second result we show that for every linear isometry (not necessarily multiplicative) $\varphi : A \to M_n(\mathbb{C})$ there exist two $n \times n$ unitary matrices $U$ and $V$ such that $\varphi(A) = UAV$ for every $A \in \mathcal{A}$. The proofs use a mix of algebra, matrix analysis, and operator theory.

Throughout the paper, we will use the symbol $S$ to denote the $n \times n$ nilpotent Jordan block of rank $n-1$:

$$S = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{bmatrix} \tag{1}$$

(here, empty spaces mean zero entries). The Toeplitz matrix algebra $\mathcal{A}$ consists of all matrices of the form $f(S)$, where $f$ is an arbitrary complex polynomial. As $\mathcal{A}$ contains the identity matrix $I$, the Toeplitz matrix algebra is a unital operator algebra. The vector subspace $\mathcal{T} = \mathcal{A} + \mathcal{A}^*$ of $M_n(\mathbb{C})$ consists of all $n \times n$ Toeplitz matrices; because $\mathcal{T}$ contains the identity and is closed under the adjoint map $X \mapsto X^*$, the space $\mathcal{T}$ is an operator system [26, Chapter 2].

The norm $\|A\|$ of $A \in M_n(\mathbb{C})$ is given by $\|A\| = \max\{\|Ax\|, : x \in \mathbb{C}^n, \|x\| = 1\}$, where the norm of $x \in \mathbb{C}^n$ is the Hilbert space norm $\|x\| = \langle x, x\rangle^{1/2}$ and where $\langle\cdot,\cdot\rangle$ is the standard inner product on $\mathbb{C}^n$. In contrast to the situation for Toeplitz operators acting on the Hardy space $H^2(\mathbb{T})$, the exact determination of the norm of a Toeplitz matrix is difficult, even in the case of $n = 2$.

If $\mathcal{L} \subset \mathcal{B}(\mathcal{H})$ is a subspace, then a linear map $\varphi : \mathcal{L} \to \mathcal{B}(\mathcal{K})$ is said to be completely contractive if the linear map $\varphi^{(k)} = \varphi \otimes \mathrm{id}_{M_k(\mathbb{C})} : \mathcal{L} \otimes M_k(\mathbb{C}) \to \mathcal{B}(\mathcal{K}) \otimes M_k(\mathbb{C})$ is contractive for every $k \in \mathbb{N}$, and is completely isometric if every $\varphi^{(k)}$ is an isometry. (Here, $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{K})$ are the algebras of bounded linear operators acting on complex Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$.) The map $\varphi$ is completely bounded if there is a $R > 0$ such that $\|\varphi^{(k)}\| \leq R$ for all $k \in \mathbb{N}$. If $\mathcal{L}$ contains the identity of $\mathcal{B}(\mathcal{H})$, then $\mathcal{L}$ is called a unital operator space, and if a unital operator space $\mathcal{L}$ is closed under the adjoint map, then $\mathcal{L}$ is called an operator system. Linear maps $\mathcal{L}_1 \to \mathcal{L}_2$ of unital operator spaces that send the identity of $\mathcal{L}_1$ to the identity of $\mathcal{L}_2$ are said to be unital.