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Complex vector lattices via functional completions

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A B S T R A C T

We show that the Fremlin tensor product $C(X)\bar{\otimes}C(Y)$ is not square mean complete when X and Y are uncountable metrizable compact spaces. This motivates the definition of complexification of Archimedean vector lattices, the Fremlin tensor product of Archimedean complex vector lattices, and a theory of powers of Archimedean complex vector lattices.

1. Introduction

The standard references for the theory of vector lattices and Banach lattices (see [14,16,21,26]) all devote some attention to complex vector lattices and complex Banach lattices, but a reading of the treatment makes one feel that something is amiss. The existence of a real cone in a complex vector space did show early promise, and in fact, is essential at times in topics ranging from spectral theory and vector measures to harmonic analysis. The emerging idea of a complex modulus in the vector space complexification $E+iE$ of a Banach lattice E dates to a 1963 paper by Rieffel (see [19] and also [20]) dealing with complex AL-spaces. In 1968 (see [13]), Lotz defined, more generally, for Banach lattices E the modulus $|f+ig|$ of an element $f+ig \in E+iE$ by

$$|f+ig| = \sup\{f\cos\theta + g\sin\theta : 0 \le \theta \le 2\pi\}. \tag{$*$}$$

Luxemburg and Zaanen extend formula $(*)$ above to all uniformly complete vector lattices (in [15]) in 1971, while studying order bounded maps and integral operators. They realized that a theory of vector lattices over $\mathbb{C}$ had to include a complex version of the Kantorovich formula for the modulus of operators in the space of order bounded operators $E \to F$, denoted by $\mathcal{L}_b(E,F)$, when E and F are Archimedean vector

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lattices, and F is Dedekind complete. That very building block was provided in 1973 by de Schipper in [24], with the existence of the supremum in $(*)$ as a condition on E and Dedekind completeness of F as follows. By defining a space of complex order bounded operators $\mathcal{L}_b(E+iE, F+iF)$, de Schipper proved that

$$\mathcal{L}_b(E,F) + i\mathcal{L}_b(E,F) = \mathcal{L}_b(E+iE, F+iF).$$

Using the subscript $\mathbb{C}$ for the complexification of a vector space, he thus proved that

$$\mathcal{L}_b(E_{\mathbb{C}}, F_{\mathbb{C}}) = \mathcal{L}_b(E,F)_{\mathbb{C}}.$$

Interestingly, Luxemburg and Zaanen had proved the complex Kantorovich formula in the earlier paper [15], mentioned above, under the stronger condition that E is uniformly complete. Schaefer, in his book [21], defines complex vector lattices axiomatically and derives formula $(*)$, but includes uniform completeness in the axioms as well.

In spite of the validity of de Schipper's theorem under the mere assumption of $(*)$, the assumption of uniform completeness has proliferated in studies on complex vector lattices, almost invariably identified with complexifications $E+iE$ of uniformly complete vector lattices E. The choice of definition for complex vector lattices in [21] as well as the standard assumption of uniform completeness in results for complex vector lattices in [26] appears to have codified that practice.

However, an alternative does exist in the literature, though it has hardly been used. Indeed, Mittelmeyer and Wolff in 1974 (see [17]) define what we call Archimedean vector lattices over $\mathbb{C}$ by axiomatizing an Archimedean modulus and they show that the resulting Archimedean complex vector lattices are exactly the ones that are vector space complexifications of Archimedean vector lattices with property $(*)$. In light of the history sketched above, their complex Archimedean vector lattices provide a ready made utility. The reader might well ask: Why then write this paper?

One answer simply is this. Rewriting all the theory for results that are valid in Archimedean real vector lattices and Archimedean complex vector lattices alike, seems a rather Herculean, and at times, uninteresting task. We hasten to add that fundamental results for real vector lattices exist that are not valid for complex ones. An example is the Riesz decomposition property (see [25]). In the opposite direction, Kalton recently (see [11]) proved surprising results for complex Banach lattices that fail for real Banach lattices. In between there is a large body of results that both theories have in common. But, even with complex vector lattices satisfactorily defined in [17], these results that are in common, lack a proper transfer mechanism, a more or less mechanical procedure that transfers real results into their complex analogues, like de Schipper's result above.

In Theorem 3.3 of this paper, we present exactly such a mechanism. We do this in three ways. First, we construct a vector lattice complexification for every Archimedean real vector lattice, moving away from the vector space complexification for which one needs to know a priori that one deals with a vector lattice in which formula $(*)$ is valid. Secondly, we prove that these new complexifications are precisely the Archimedean vector lattices over $\mathbb{C}$ introduced by Mittelmeyer and Wolff. Thirdly, we show that these newly constructed complexifications satisfy a natural universal property which, in many instances, tremendously facilitates the transfer mechanism from real results to complex results. We introduce this vector lattice complexification with a purpose in mind: differentiation in Archimedean vector lattices over $\mathbb{C}$ via multilinear maps and tensor products. The real version of such differentiation in vector lattices was introduced by Loane in [12]. A rapid development of polynomials on vector lattices is currently under way and complex tensor products and powers of complex vector lattices are needed. Motivated initially by this attempt at complex differentiation, we started by looking at the real Fremlin tensor product $E\bar{\otimes}E$ and were willing to assume uniform completeness of E, which has been the modus operandi in the literature, in order for

$$E\bar{\otimes}E + i(E\bar{\otimes}E)$$