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Stability analysis in magnetic resonance elastography II

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ABSTRACT

We consider the inverse problem of finding unknown elastic parameters from internal measurements of displacement fields for tissues. In the sequel to [5], we use pseudodifferential methods for the problem of recovering the shear modulus for Stokes systems from internal data. We prove stability estimates in $d = 2, 3$ with reduced regularity on the estimates and show that the presence of a finite dimensional kernel can be removed. This implies the convergence of the Landweber numerical iteration scheme. We also show that these hypotheses are natural for experimental use in constructing shear modulus distributions.

1. Introduction

This article uses pseudodifferential methods to sharpen recent stability estimates for an inverse problem of Magnetic Resonance Elastography (MRE), with short proofs. The new proofs allow to analyze practical numerical reconstruction methods.

In Magnetic Resonance Elastography, internal measurements of time-harmonic displacement fields offer the possibility of a highly resolved reconstruction of shear modulus distributions. It is motivated by the detection of cancerous anomalies in their early stages [1]. See [3,6,7,10,11,13–15] for relevant related works.

The reduced regularity assumptions of the stability estimates in this article prove relevant for numerical reconstruction schemes. We show how they directly relate to classical results for overdetermined elliptic boundary problems and their analysis using pseudodifferential operators. The analysis is based on a Stokes system as in [5].

Let Ω denote a simply-connected bounded domain in $\mathbb{R}^d$ where $d = 2, 3$ with $\mathcal{C}^\infty$-boundary $\partial\Omega$. We consider the following boundary value problem for the elasticity equations

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$$\begin{cases} \nabla(\lambda(x)\nabla \cdot u_\lambda) + \omega^2 u_\lambda(x) + 2\nabla \cdot \mu(x)\nabla^s u_\lambda(x) = 0 \quad \text{in } \Omega, \\ u_\lambda(x) = F(x) \quad \text{on} \quad \partial\Omega \ , \end{cases} \tag{1.1}$$

where

$$2\nabla^s u_\lambda = \nabla u_\lambda + (\nabla u_\lambda)^t$$

and ∇u_λ is the matrix $(\partial_j u_{\lambda,i})_{i,j=1}^d$, with $u_{\lambda,i}$ the i-th component of u_λ. The Lamé coefficients (respectively, the shear and the compressional modulus) $\lambda, \mu \in \mathcal{C}^1(\bar{\Omega})$ satisfy

$$\lambda \geq \lambda_{min} = \min\{\lambda(x) : x \in \bar{\Omega}\} > 0, \tag{1.2}$$

$$\mu \geq \mu_{min} = \min\{\mu(x) : x \in \bar{\Omega}\} > 0, \tag{1.3}$$

and for simplicity we assume $\int\limits_{\partial\Omega} F(x) \cdot \nu(x)\, dx = 0$.

If $F \in H^{1/2}(\partial\Omega)$ and (1.2) and (1.3) are satisfied, there exists a unique solution $u_\lambda \in H^1(\Omega)^d$ to (1.1) even for $\lambda, \mu \in L^\infty(\Omega)$. In particular, $\nabla^s u_\lambda \in L^2(\Omega)^d$. Higher regularity $\nabla^s u_\lambda \in H^4(\Omega)^d$ holds under the additional assumption that $\lambda, \mu \in \mathcal{C}^4(\bar{\Omega})$, $F \in H^{9/2}(\partial\Omega)^d$.

In [5] it was shown that for $2\mu_{\max} < 3\lambda_{\min}$ the solution $u_\lambda(x)$ is approximated in $H^1(\Omega)^d$ norm up to $\mathcal{O}(\lambda^{-1/2})$ by the solution to the Stokes problem

$$\begin{cases} \omega^2 u(x) + 2\nabla \cdot \mu(x)\nabla^s u(x) + \nabla p(x) = 0 \quad \text{in } \Omega, \\ \nabla \cdot u(x) = 0 \quad \text{in} \quad \Omega, \\ u(x) = F(x) \quad \text{on} \quad \partial\Omega, \\ \int\limits_\Omega p(x)\, dx = 0, \quad \int\limits_{\partial\Omega} F(x) \cdot \nu(x)\, dx = 0. \end{cases} \tag{1.4}$$

If we examine the solutions in 2 and 3 dimensions, we find we can reconstruct a displacement of the shear modulus μ in a stable way from the difference of the solutions. Hybrid modalities involve exciting the system with more than one wave or modality. In 3d it was already shown in [5] using elliptic regularity theory that hybrid modalities are necessary for reconstruction of the shear modulus. This necessity is in contrast to dimension 2, where one modality is sufficient. Technically, one takes a curl of the equation (1.4), and the different properties of the curl in 2 resp. 3 dimensions lead to an elliptic (2d) resp. injectively elliptic (3d) problem for the reconstruction.

The goal of this article is to show how pseudodifferential operators and the theory of elliptic boundary problems leads to simple proofs of stronger stability estimates for the inverse problem. The estimates reduce the necessary order of regularity and allow the analysis of practical numerical reconstruction methods.

In Section 2 we introduce basic notions of elliptic boundary problems and the Shapiro–Lopatinskii ellipticity condition. A basic Fredholm theorem is recalled in Theorem 1. In Sections 3 and 4 we apply Theorem 1 to conclude basic stability estimates (Theorems 3 and 2) in the presence of a finite dimensional kernel. Refined stability estimates in L^2, based on stronger assumptions, are the content of Section 5. We also show that locally it is possible to remove the presence of the finite dimensional kernel under certain hypotheses on the system, Corollaries 3 and 4. These results imply convergence of the numerical Landweber iteration scheme, which is discussed in Section 8. We also show that the assumptions which are made on the symbols are natural for numerical experiments in Section 7.

Notation. In this paper we use the Einstein summation convention. For two matrices A and B, the inner product is denoted by

$$A : B = a_{ij} b_{ji},$$