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On complex symmetric Toeplitz operators ☆

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ABSTRACT

In this paper, we give a characterization of a complex symmetric Toeplitz operator T_φ on the Hardy space H^2. Moreover, if T_φ is a complex symmetric Toeplitz operator, we provide a necessary and sufficient condition for T_φ to be normal. Finally, we investigate these T_φ with finite symbols.

1. Introduction

Let $\mathcal{L}(\mathcal{H})$ be the algebra of bounded linear operators on a separable complex Hilbert space $\mathcal{H}$. For an operator $T \in \mathcal{L}(\mathcal{H})$, let T^* denote the adjoint of T. An operator $T \in \mathcal{L}(\mathcal{H})$ is *normal* if $T^*T = TT^*$, *subnormal* if there exists a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a normal operator N on $\mathcal{K}$ such that $N\mathcal{H} \subset \mathcal{H}$ and $T = N|_{\mathcal{H}}$, and *hyponormal* if $T^*T - TT^* \geq 0$.

A *conjugation* on $\mathcal{H}$ is an antilinear operator $C : \mathcal{H} \to \mathcal{H}$ with $C^2 = I$ which satisfies $\langle Cx, Cy\rangle = \langle y, x\rangle$ for all $x, y \in \mathcal{H}$. For a conjugation C, there exists an orthonormal basis $\{e_n\}_{n=0}^{\infty}$ for $\mathcal{H}$ such that $Ce_n = e_n$ for all n (see [4] and [6] for more details). We call an operator $T \in \mathcal{L}(\mathcal{H})$ *complex symmetric* if there exists a conjugation C on $\mathcal{H}$ such that $T = CT^*C$. The class of complex symmetric operators includes all normal operators, Hankel operators, truncated Toeplitz operators, and Volterra integration operators. We refer the reader to [6,7,12,11] for more details, including historical comments and references.

Let L^2 be the Lebesgue (Hilbert) space on the unit circle $\partial\mathbb{D}$, and let L^∞ be the Banach space of all essentially bounded functions on $\partial\mathbb{D}$. Then it is well-known that $\{e_n(z) = z^n : n = 0, \pm1, \pm2, \pm3, \ldots\}$ is an

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orthonormal basis for L^2. If $f \in L^2$, then the function f is expressed as $f(z) = \sum_{n=-\infty}^{\infty} \hat{f}(n) z^n$ where $\hat{f}(n)$ denotes the nth Fourier coefficient of f. The Hilbert Hardy space, denoted by H^2, consists of all functions f analytic on the open unit disk $\mathbb{D}$ with the power series representation

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ where } \sum_{n=0}^{\infty} |a_n|^2 < \infty,$$

or equivalently, with $\sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \right) < \infty$. It is clear that $H^2 = \overline{\text{span}\{z^n : n = 0, 1, 2, 3, \ldots\}}$.

For any $\varphi \in L^\infty$, the *Toeplitz operator* $T_\varphi : H^2 \to H^2$ is defined by the formula

$$T_\varphi f = P(\varphi f)$$

for $f \in H^2$ where P denotes the orthogonal projection of L^2 onto H^2. It is known that T_φ is bounded if and only if $\varphi \in L^\infty$ and, in which case, $\|T_\varphi\| = \|\varphi\|_\infty$. A Toeplitz operator T_φ is called *analytic* if $\varphi \in H^\infty$, i.e., φ is a bounded analytic function on the unit disc $\mathbb{D}$ and *coanalytic* if $\overline{\varphi} \in H^\infty$ where $\overline{\varphi}$ denotes the complex conjugate of φ. For $\lambda \in \mathbb{D}$, the *reproducing kernel* K_λ for H^2 is given by $K_\lambda(z) := \frac{1}{1-\overline{\lambda} z}$ and $\langle f, K_\lambda \rangle = f(\lambda)$ for all $f \in H^2$.

The study of complex symmetric operators and Toeplitz operators provides deep and important connections with various problems in the field of quantum mechanics (see [5,13,1]). In 1960s, A. Brown and P. Halmos [2] proved that T_φ is normal if and only if $\varphi = \alpha + \beta\rho$ where ρ is a real-valued function in L^∞ and $\alpha, \beta \in \mathbb{C}$. In 1970, P. Halmos [9] raised for the problem of characterizing subnormal Toeplitz operators. It is apparent that every normal or analytic T_φ is subnormal. In general, T_φ may not be a complex symmetric operator. However, if T_φ is a complex symmetric and hyponormal operator, then T_φ is normal from [14]. Recently, K. Guo and S. Zhu [8] have raised the following interesting question.

Question. Characterize a complex symmetric Toeplitz operator on the Hardy space H^2 of the unit disk.

In this paper, we study properties of complex symmetric Toeplitz operators T_φ on the Hardy space H^2. In Section 2, we provide a characterization of such an operator T_φ and we give a necessary and sufficient condition for a complex symmetric Toeplitz operator T_φ to be normal. In Section 3, we examine these T_φ with finite symbols.

2. Complex symmetric Toeplitz operators

In this section, we study complex symmetric Toeplitz operators T_φ on the Hardy space H^2. In particular, we give a characterization of such operators. We first start with the following theorem.

Theorem 2.1. *For $\varphi \in L^\infty$, let T_φ be a complex symmetric operator on H^2. If T_φ is analytic or coanalytic, then φ is either identically zero on $\mathbb{D}$ or a nonzero constant function on $\mathbb{D}$.*

Proof. Let φ be not identically zero on $\mathbb{D}$ and let T_φ be analytic. If $\varphi(\lambda) = 0$ for some λ in $\mathbb{D}$, then $\varphi(z) \neq 0$ for all z in some open set U of $\mathbb{D}$ which does not contain λ. Since $T_\varphi^* K_\lambda = \overline{\varphi(\lambda)} K_\lambda = 0$, $\|T_\varphi C K_\lambda\| = \|T_\varphi^* K_\lambda\| = 0$, and so $T_\varphi C K_\lambda(z) = 0$ for all $z \in \mathbb{D}$. Moreover, since $\varphi \in H^\infty$, $\varphi(z) C K_\lambda(z) = 0$ for all $z \in \mathbb{D}$. This means that $C K_\lambda(z) = 0$ for all $z \in U$ and hence $C K_\lambda \equiv 0$ on $\mathbb{D}$ by the identity theorem, which is a contradiction. Hence φ does not vanish on $\mathbb{D}$.

Now fix $\alpha \in \mathbb{D}$. Then

$$C T_\varphi^* K_\alpha - T_\varphi C K_\alpha = C \overline{\varphi(\alpha)} K_\alpha - T_\varphi C K_\alpha$$