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Existence and construction of quasi-stationary distributions for one-dimensional diffusions

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A B S T R A C T

In this paper, we study quasi-stationary distributions (QSDs) for one-dimensional diffusions killed at 0, when 0 is a regular boundary and $+\infty$ is a natural boundary. More precisely, we not only give a necessary and sufficient condition for the existence of a QSD, but we also construct all QSDs for the one-dimensional diffusions. Moreover, we give a sufficient condition for R-positivity of the process killed at the origin. This condition is only based on the drift, which is easy to check.

1. Introduction

We are interested in the long-term behavior of killed Markov processes. Conditional stationarity, which we call quasi-stationarity, is one of the most interesting topics in this direction. For quasi-stationary distribution (QSD), we know that the study of QSDs is a long standing problem in several areas of probability theory and a complete understanding of the structure of QSDs seems to be available only in rather special situations such as Markov chains on finite sets or more general processes with compact state space. The main motivation of this work is the existence and construction of QSDs for one-dimensional diffusion X killed at 0, when 0 is a regular boundary and $+\infty$ is a natural boundary. Moreover, we give a sufficient condition in order for the process X killed at 0 to be R-positive.

To the best of our knowledge, Mandl [11] is the first one to study the existence of a QSD for continuous time diffusion process on the half line. If Mandl's conditions are satisfied, the existence of the Yaglom limit and that of a QSD for killed one-dimensional diffusion processes have been proved by various authors (see, e.g., [4,8,10,12,17]). If Mandl's conditions are not satisfied, Cattiaux et al. who studied the existence and uniqueness of the QSD for one-dimensional diffusions killed at 0 and whose drift is allowed to go to $-\infty$ at 0 and the process is allowed to have an entrance boundary at $+\infty$, have done a pioneering work (see [1]). In

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this case, under the most general conditions, Littin proves the existence of a unique QSD and of the Yaglom limit in [9], which is closely related to [1]. Although [4,8,10,12,15,17] and [1] make the key contributions, the structure of QSDs of killed one-dimensional diffusions has not been completely clarified. This leads us to further study QSDs for one-dimensional diffusions.

Another notion is R-positivity, which, in general, is not easy to check, is sufficient to facilitate the straightforward calculation of QSDs for a process from the eigenvectors, eigenmeasures and eigenvalues of its transition rate matrix. The classification of killed one-dimensional diffusions has been studied by Martínez and San Martín [15]. They gave necessary and sufficient conditions, in terms of the bottom eigenvalue function, for R-recurrence and R-positivity of one-dimensional diffusions killed at the origin.

In this paper, the main novelty is that we not only give a necessary and sufficient condition for the existence of a QSD, but we also construct all QSDs for one-dimensional diffusion X killed at 0, when 0 is a regular boundary and $+\infty$ is a natural boundary. Moreover, compared with [15], we give an explicit criterion for the process X killed at 0 is R-positive.

The remainder of this paper is organized as follows. In Section 2 we present some preliminaries that will be needed in the sequel. In Section 3 we characterize all QSDs for one-dimensional diffusion X killed at 0, when 0 is a regular boundary and $+\infty$ is a natural boundary. In Section 4 we mainly show under what direct conditions on the drift the process is R-positive. We conclude in Section 5 with some examples.

2. Preliminaries

We consider the generator $Lu = \frac{1}{2}\partial_{xx}u - q\partial_x u$. Denote by X the diffusion whose infinitesimal generator is L, or in other words the solution of the stochastic differential equation (SDE)

$$dX_t = dB_t - q(X_t)dt, \qquad X_0 = x > 0, \tag{2.1}$$

where $(B_t; t \geq 0)$ is a standard one-dimensional Brownian motion and $q \in C^1([0,\infty))$. Thus, $-q$ is the drift of X. Observe that, under the condition $q \in C^1([0,\infty))$, $\int_0^d e^{Q(y)}dy < \infty$ and $\int_0^d e^{-Q(y)}dy < \infty$ for some (and, therefore, for all) $d > 0$, which is equivalent to saying that the boundary point 0 is regular in the sense of Feller, where $Q(y) = \int_0^y 2q(x)dx$.

Let $\mathbb{P}_x$ and $\mathbb{E}_x$ stand for the probability and the expectation, respectively, associated with X when initiated from x. For any probability measure ν on $(0,\infty)$, we denote by $\mathbb{P}_\nu$ the probability associated with the process X initially distributed with respect to ν. Let $\tau_a := \inf\{t > 0 : X_t = a\}$ be the hitting time of a. We are mainly interested in the case $a = 0$ and we denote $\tau = \tau_0$. As usual X^τ corresponds to X killed at 0.

Associated with q, we consider the function

$$\Lambda(x) = \int_0^x e^{Q(y)}dy. \tag{2.2}$$

Notice that Λ is the scale function for X. It satisfies $L\Lambda \equiv 0$, $\Lambda(0) = 0$, $\Lambda'(0) = 1$. It will be useful to introduce the following measure defined on $(0,\infty)$:

$$\mu(dy) := e^{-Q(y)}dy. \tag{2.3}$$

Notice that μ is the speed measure for X.

Let $L^* = \frac{1}{2}\partial_{xx} + \partial_x(q\cdot)$ be the formal adjoint operator of L. We denote by φ_λ the solution of

$$L^*\varphi_\lambda = -\lambda\varphi_\lambda, \quad \varphi_\lambda(0) = 0, \quad \varphi'_\lambda(0) = 1, \tag{2.4}$$